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On the BCF-Characters of Complex Manifolds with Kähler Metrics of Constant Scalar Curvature.

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Abstract: In this paper using BCF-character, we give examples of complex manifolds which do not admit any Kähler metric of constant scalar curvature whose Kähler form is contained in an assigned Hodge class. We also give a restrictive condition on the fixed point data of a cyclic action on a complex manifold which admits a Kähler metric of constant scalar curvature. Our main results are Theorem 2.1 and Theorem 3.1.

Key words: BCF-character, complex manifolds, Kähler metric, constant scalar curvature, fixed point set

1. Introduction

Let M be a closed n-dimensional manifold with a Riemannian metric $g = \left(g_{ij}\right)$ with respect to a local coordinate $(x^1, \cdots, x^n)$. Then the scalar curvature R of the Riemannian metric is defined by

$$R = \sum_{k,\ell,j=1}^{n} g^{j\ell}\left(\frac{\partial \Gamma^k_{\ell j}}{\partial x^k} - \frac{\partial \Gamma^k_{kj}}{\partial x^\ell} + \sum_{m=1}^{n}\left(\Gamma^m_{\ell j}\Gamma^k_{km} - \Gamma^m_{kj}\Gamma^k_{\ell m}\right)\right)$$

where $\left(g^{ij}\right)$ is the inverse matrix of $\left(g_{ij}\right)$ and Γ^i_{jk} is the Christoffel's symbol defined by

$$\Gamma^i_{jk} = \frac{1}{2}\sum_{\ell=1}^{n} g^{i\ell}\left(\frac{\partial g_{\ell j}}{\partial x^k} + \frac{\partial g_{\ell k}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\ell}\right).$$

Scalar curvature is a fundamental quantity in differential geometry, and is also a fundamental quantity in general relativity when $n = 4$. The question of whether a manifold admits a Riemannian metric of constant scalar curvature or not, namely, the question of whether a solution $\left(g_{ij}\right)$ of the nonlinear partial differential equation R =constant exists or not is a classical problem, and has stimulated active research. Though several topological obstructions to the existence of the metric of positive constant scalar curvature had been found (see [6]), in [5] Kazdan and Warner proved that any closed manifold admits a metric of negative constant scalar curvature. But if the closed manifold is a complex manifold and the Riemannian metric is a Kähler metric, namely, a Riemannian metric which is compatible with the complex structure, the result of Kazdan-Warner does not hold.

Let M be an m-dimensional compact complex manifold, Aut(M) the Lie group consisting of all biholomorphic automorphisms of M and H(M) its Lie algebra consisting of all holomorphic vector fields on M. Note that the real dimension of M is $2m$. In [7], [8], Lichnerowicz proved that a complex manifold does not admit any Kähler metric of constant scalar curvature unless the Lie algebra H(M) has a specific structure. But it is in general difficult to determine the structure of the Lie algebra H(M) and the Lichnerowicz's result asserts nothing about the existence of the Kähler metric of constant scalar curvature when the Lie algebra H(M) has the specific structure.

In [1], [2], [3] an obstruction to the existence of a Kähler metric on M of constant scalar curvature whose Kähler form is contained in an assigned Hodge class is obtained, in [9] the obstruction is lifted to a Lie group character, and in [4] a formula to calculate the character is obtained.

Let $c_1(M) \in \mathrm{H}^2(M;\mathbb{Z})$ be the first Chern class of the tangent bundle TM, $\Omega \in \mathrm{H}^2(M;\mathbb{Z})$ a Hodge class and $[M]$ the fundamental cycle of M. Then a number μ_Ω is defined by

$$\mu_\Omega = \frac{(\Omega^{m-1}c_1(M))[M]}{\Omega^m[M]}$$

where the product of elements of $\mathrm{H}^2(M;\mathbb{Z})$ is the cup product. Let G be a compact connected subgroup of Aut(M). Then the obstruction of Bando-Calabi-Futaki is a Lie group character

$$\widehat{f_\Omega} : G \longrightarrow \mathbb{C}/(\mathbb{Z} + \mu_\Omega\mathbb{Z}),$$

which is called the BCF character.

Thorem 1.1 ([1],[2],[3],[9]). *If M admits a Kähler metric of constant scalar curvature whose Kähler form is contained in Ω, then $\widehat{f_\Omega}$*

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vanishes, namely, $\widehat{f_\Omega}(\sigma) \in \mathbb{Z} + \mu_\Omega \mathbb{Z}$ *for any* $\sigma \in G$.

Now let G_0 denote the dense subset of G consisting of elements of finite order. Then we can define a map $F_L : G_0 \longrightarrow \mathbb{C}$ by using the fixed point data of the G-action as follows. Let σ be an element of G_0 which has a finite order p and $S(k)$ the fixed point set of σ^k, which consists of compact connected complex submanifolds N of M. Let $\nu(N, M)$ be the normal bundle of N in M and α the primitive p-th root of unity. Then $\nu(N, M)$ is decomposed into the direct sum of subbundles

$$\nu(N, M) = \oplus_j V_j\left(\alpha^{\ell_j}\right)$$

where σ acts on $V_j\left(\alpha^{\ell_j}\right)$ via multiplication by α^{ℓ_j}. We define the characteristic class $\Phi(\nu(N, M))$ by

$$\Phi(\nu(N, M)) = \prod_j \prod_{k=1}^{r_j} \frac{1}{1 - \alpha^{-k\ell_j} e^{-x_k}} \in H^*(N; \mathbb{C}) \quad \left(r_j = \operatorname{rank}_{\mathbb{C}}\left(V_j\left(\alpha^{\ell_j}\right)\right)\right)$$

where $\prod_k (1 + x_k)$ is the total Chern class of $V_j\left(\alpha^{\ell_j}\right)$. Assume that the action of σ preserves the Hodge class Ω and that the σ-action lifts to an action on a holomorphic line bundle L with $c_1(L) = \Omega$. Suppose that $\sigma|(K_M|N) = \alpha^\kappa$ (namely, σ acts on $K_M|N$ via multiplication by α^κ) and $\sigma|(L|N) = \alpha^\rho$ for $\kappa, \rho \in \mathbb{Z}$. Let $\mathrm{Td}(TN)$ be the Todd class of TN and $[N]$ the fundamental cycle of N. N, $V_j\left(\alpha^{\ell_j}\right)$, κ, ρ are called the fixed point data of σ. Then $F_L(\sigma)$ is defined by the fixed point data as follows.

Definition 1.2.

$$\begin{aligned} F_L(\sigma) &= (m+1) \sum_{j=0}^{m} (-1)^j \binom{m}{j} \left\{ S^{-1}(m-2j) - S^{+1}(m-2j) \right\} \\ &\quad - m\mu_\Omega \sum_{j=0}^{m+1} (-1)^j \binom{m+1}{j} S^0(m+1-2j) \qquad (\mathrm{mod}\ \mathbb{Z} + \mu_\Omega \mathbb{Z}) \end{aligned}$$

where for $\varepsilon = 1, 0, -1$

$$S^\varepsilon(n) = \sum_{N \subset S(k)} S_N^\varepsilon(n),$$

$$S_N^\varepsilon(n) = \frac{1}{p} \sum_{k=1}^{p-1} \frac{1}{1 - \alpha^k} (\alpha^{k(\varepsilon\kappa + n\rho)} e^{c_1(K_M^\varepsilon|N) + n c_1(L|N)} - 1)^{m+1} \mathrm{Td}(TN) \Phi(\nu(N, M))[N].$$

Assumption 1.3. *Assume that the fixed point set* $S(k)$ *is independent of* k *and every connected component* N *has a cell decomposition with no codimension one cells.*

For example the fixed point set $S(k)$ is independent of k if the order p is a prime number and N has a cell decomposition with no codimension one cells if N is a complex projective space.

Thorem 1.4 ([4]). *Under Assumption 1.3 the following equality holds:*

$$\widehat{f_\Omega}(\sigma) \equiv F_L(\sigma) \qquad (\mathrm{mod}\ \mathbb{Z} + \mu_\Omega \mathbb{Z}).$$

Remark 1.5. *As we see in* [4], *Theorem 1.4 holds without Assumption 1.3 when* $L = (K_M^{-1})^\lambda$ *for some integer* λ.

Using Theorem 1.4 above, we can show that certain complex manifolds M does not admit any Kähler metric of constant scalar curvature whose Kähler form is contained in an assigned Hodge class without determining the structure of the Lie algebra $\mathrm{H}(M)$. In section 2, we show that one or two points blow-up of $\mathbb{CP}^2$ does not admit any Kähler metric of constant scalar curvature whose Kähler form is contained in an integral multiple of the first Chern class. Theorem 1.4 can be used as the restrictive condition on the fixed point data. In section 3, using Theorem 1.4, we show that the fixed point data of the action of the circle group S^1 on the product manifold of complex projective spaces have to satisfy a certain condition.

2. BCF-characters associated to an integral multiple of the first Chern class

In this section using Theorem 1.4, we prove that certain complex manifolds do not admit any Kähler metric of constant scalar curvature whose Kähler form is contained in an integral multiple of the first Chern class $c_1(M)$. Here let λ be any non-zero integer and set $L = (K_M^{-1})^\lambda$. Then any action on M lifts to an action on L. Since $\Omega = c_1(L)$ is equal to $\lambda c_1(M)$, we have

$$\mu_\Omega = \frac{(\lambda c_1(M) c_1(M))[M]}{(\lambda c_1(M))^2[M]} = \frac{1}{\lambda}.$$

Let M_k denote the k points blow-up of $\mathbb{CP}^2$. Then we have the next theorem.

Thorem 2.1. *If $k = 1, 2$, $Aut(M_k)$ contains S^1 and there exists an element $\sigma \in S^1$ which satisfies $\widehat{f_\Omega}(\sigma) \neq 0 \in \mathbb{C}/\mathbb{Z}$.*

Proof. First let M_1 be the surface obtained from $\mathbb{CP}^2$ by blowing up one point $[1 : 0 : 0]$ where $[z_0 : z_1 : z_2]$ is the homogeneous coordinate on $\mathbb{CP}^2$. Let $\mathrm{D}(a, b, c)$ denote the diagonal matrix with diagonal entries a, b, c. Let σ be an element of $\mathrm{Aut}(M_1)$ which is naturally induced by the element of $\mathrm{Aut}(\mathbb{CP}^2) = PGL(3; \mathbb{C})$ represented by $\mathrm{D}(1, \alpha, \alpha)$ where α is the primitive p th root of unity for odd prime p. Then the action of $\mathbb{Z}_p = \langle\sigma\rangle$ lifts to an action of S^1 and the fixed point set of σ^k $(1 \le k \le p-1)$ is equal to the disjoint union of the exceptional divisor E over $[1 : 0 : 0]$ and the hyperplane H defined by $z_0 = 0$, which have cell decompositions with no codimension one cells. The normal bundle $\nu(E, M)$ is equal to the tautological line bundle J and the normal bundle $\nu(H, M)$ is equal to its dual J^*, σ acts on J via multiplication by α and on J^* via multiplication by α^{-1} and hence the following equalities hold. (See [10].)

$$
\begin{aligned}
&Td(TE) = 1 + u\,,\ Td(TH) = 1 + v\,,\quad c_1(\nu(E, M)) = -u\,,\ c_1(\nu(H, M)) = v \\
&c_1(K_M^{-1}|E) = c_1(TE) + c_1(\nu(E, M)) = u\,,\quad c_1(K_M^{-1}|H) = c_1(TH) + c_1(\nu(E, H)) = 3v\,,\\
&c_1(L|E) = \lambda c_1(K_M^{-1}|E) = \lambda u\,,\quad c_1(L|H) = \lambda c_1(K_M^{-1}|H) = 3\lambda v\,,\\
&\sigma^k|\nu(E, M) = \sigma^k|(K_M^{-1}|E) = \alpha^k\,,\quad \sigma^k|\nu(E, H) = \sigma^k|(K_M^{-1}|H) = \alpha^{-k}\,,\\
&\sigma^k|(L|E) = \alpha^{k\lambda}\,,\quad \sigma^k|(L|H) = \alpha^{-k\lambda}\,,\\
&\Phi(\nu(E, M)) = \frac{1}{1 - \alpha^{-k}e^u}\,,\quad \Phi(\nu(H, M)) = \frac{1}{1 - \alpha^k e^{-v}}
\end{aligned}
$$

where u, v are positive generators of $\mathrm{H}^2(E; \mathbb{Z}) = \mathbb{Z}$, $\mathrm{H}^2(H; \mathbb{Z}) = \mathbb{Z}$ respectively such that $u^k[E] = v^k[H] = \delta_{k1}$ where $[E]$, $[H]$ denote the fundamental cycles. Hence, for $\varepsilon = 1, 0, -1$ we have

$$
\begin{aligned}
S^\varepsilon(n) &= \frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{1-\alpha^k}\left(\alpha^{(\lambda n-\varepsilon)k}e^{(\lambda n-\varepsilon)u} - 1\right)^3(1+u)\frac{1}{1-\alpha^{-k}e^u}[E] \\
&\quad + \frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{1-\alpha^k}\left(\alpha^{(-\lambda n+\varepsilon)k}e^{3(\lambda n-\varepsilon)v} - 1\right)^3(1+v)\frac{1}{1-\alpha^k e^{-v}}[H] \\
&= \frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{1-\alpha^k}\lim_{u\to 0}\frac{\partial}{\partial u}\left\{\left(\alpha^{(\lambda n-\varepsilon)k}e^{(\lambda n-\varepsilon)u} - 1\right)^3(1+u)\frac{1}{1-\alpha^{-k}e^u}\right\} \\
&\quad + \frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{1-\alpha^k}\lim_{v\to 0}\frac{\partial}{\partial v}\left\{\left(\alpha^{(-\lambda n+\varepsilon)k}e^{3(\lambda n-\varepsilon)v} - 1\right)^3(1+v)\frac{1}{1-\alpha^k e^{-v}}\right\}
\end{aligned}
$$

Now set $\beta = \alpha^k$ and

$$
\begin{aligned}
g(\beta) &= \frac{1}{1-\beta}\lim_{u\to 0}\frac{\partial}{\partial u}\left\{\left(\beta^{\lambda n-\varepsilon}e^{(\lambda n-\varepsilon)u} - 1\right)^3(1+u)\frac{1}{1-\beta^{-1}e^u}\right\} \\
&\quad + \frac{1}{1-\beta}\lim_{v\to 0}\frac{\partial}{\partial v}\left\{\left(\beta^{-\lambda n+\varepsilon}e^{3(\lambda n-\varepsilon)v} - 1\right)^3(1+v)\frac{1}{1-\beta e^{-v}}\right\}.
\end{aligned}
$$

Then since the equality $\beta^{-m} = \beta^{p-m}$ implies that there exists an integral polynomial $\Psi(x)$ such that

$$
g(\beta) = \frac{\Psi(\beta)}{(1-\beta)^3}
$$

and the equalities

$$
\begin{aligned}
\lim_{\beta\to 1}(1+\beta)g(\beta) &= \lim_{\beta\to 1}\frac{1+\beta}{1-\beta}\lim_{u\to 0}\frac{\partial}{\partial u}\left\{\left(\beta^{\lambda n-\varepsilon}e^{(\lambda n-\varepsilon)u} - 1\right)^3(1+u)\frac{1}{1-\beta^{-1}e^u}\right\} \\
&\quad + \lim_{\beta\to 1}\frac{1+\beta}{1-\beta}\lim_{v\to 0}\frac{\partial}{\partial v}\left\{\left(\beta^{-\lambda n+\varepsilon}e^{3(\lambda n-\varepsilon)v} - 1\right)^3(1+v)\frac{1}{1-\beta e^{-v}}\right\} = 8(\lambda n-\varepsilon)^3\,, \\
\lim_{\beta\to -1}(1+\beta)g(\beta) &= \lim_{\beta\to -1}\frac{(1+\beta)\Psi(\beta)}{(1-\beta)^3} = 0
\end{aligned}
$$

hold, it follows from Lemma A in Appendix that

$$
S^\varepsilon(n) = \frac{1}{p}\sum_{k=1}^{p-1}g(\alpha^k) \equiv -\frac{4(\lambda n-\varepsilon)^3}{p} = \frac{1}{p}\left(-4\lambda^3 n^3 + 12\varepsilon\lambda^2 n^2 - 12\lambda n + 4\varepsilon\right) \qquad (\mathrm{mod}\ \mathbb{Z}).
$$

Hence it follows from Theorem 1.4 and Lemma B in Appendix that

$$
\widehat{f_\Omega}(\sigma) \equiv 3\sum_{i=0}^{2}(-1)^i\binom{2}{i}\left(S^{-1}(2-2i) - S^{+1}(2-2i)\right) - \frac{2}{\lambda}\sum_{i=0}^{3}(-1)^i\binom{3}{i}S^0(3-2i)
$$

$$= \frac{1}{p}\left(3\cdot(-12-12)\cdot\lambda^2\cdot 2^2 2! - \frac{2}{\lambda}\cdot(-4)\lambda^3\cdot 2^3 3!\right) = -\frac{192}{p}\lambda^2 \qquad (\text{mod } \mathbb{Z}).$$

Therefore $\widehat{f_\Omega}(\sigma) \neq 0$ if $p \geq 5$ and p is not a divisor of λ.

Next let M_2 be the surface obtained from $\mathbb{CP}^2$ by blowing up two points $[1:0:0]$, $[0:1:0]$. Let $\pi : M_2 \longrightarrow \mathbb{CP}^2$ be the projection and σ an element of $\mathrm{Aut}(M_2)$ which is naturally induced by the element of $\mathrm{Aut}(\mathbb{CP}^2)$ represented by $\mathrm{D}\left(1, \alpha, \alpha^2\right)$ where $\alpha = e^{2\pi i/p}$ for an odd integer $p \geq 3$. Then the action of $\mathbb{Z}_p = \langle\sigma\rangle$ lifts to an action of S^1 and the fixed point set of σ^k-action $(1 \leq k \leq p-1)$ consists of five points p_1, p_2, p_3, p_4, p_5 where $p_1 = \pi^{-1}([0:0:1])$, $p_2 \in \pi^{-1}([1:0:0])$ is the point in M_2 defined by the line:$z_1 = 0$ through the point $[1:0:0]$ in $\mathbb{CP}^2$, $p_3 \in \pi^{-1}([1:0:0])$ is the point in M_2 defined by the line:$z_2 = 0$ through the point $[1:0:0]$ in $\mathbb{CP}^2$, $p_4 \in \pi^{-1}([0:1:0])$ is the point in M_2 defined by the line:$z_0 = 0$ through the point $[0:1:0]$ in $\mathbb{CP}^2$ and $p_5 \in \pi^{-1}([0:1:0])$ is the point in M_2 defined by the line:$z_2 = 0$ through the point $[0:1:0]$ in $\mathbb{CP}^2$. Let $T_j = \sigma^k|_{\nu(p_j, M_2)}$ denote the transformation of the tangent space $T_{p_j}M_2 = \nu(p_j, M_2)$ induced by σ^k. Then we can see that $T_j = \mathrm{D}\left(\alpha^{s_j k}, \alpha^{t_j k}\right)$ $(1 \leq j \leq 5)$ where

$$(s_1, t_1) = (-2, -1)\,,\ (s_2, t_2) = (-1, 2)\,,\ (s_3, t_3) = (1, 1)\,,\ (s_4, t_4) = (-2, 1)\,,\ (s_5, t_5) = (-1, 2)\,.$$

Then σ^k acts on $K_{M_2}^{-1}|_{p_j}$ via multiplication by $\alpha^{(s_j+t_j)k}$. Since the points clearly have cell decompositions with no codimension one cells and the first Chern classes of vector bundles restricted to points vanish, we have

$$S^\varepsilon(n) = \frac{1}{p}\sum_{k=1}^{p-1} g(\alpha^k)$$

where

$$g(\beta) = \frac{1}{1-\beta}\sum_{j=1}^{5}\left(\beta^{(s_j+t_j)(\lambda n-\varepsilon)} - 1\right)^3 \frac{1}{(1-\beta^{-s_j})(1-\beta^{-t_j})}\,.$$

Then since there exists an integral polynomial $\Phi(x)$ such that

$$g(\beta) = \frac{\Phi(\beta)}{(1+\beta)(1-\beta)^3}$$

and direct computation shows that

$$\lim_{\beta\to 1}\frac{\Phi(\beta)}{(1-\beta)^3} = \lim_{\beta\to 1}(1+\beta)g(\beta) = 12(\lambda n-\varepsilon)^3\,,$$
$$\lim_{\beta\to -1}\frac{\Phi(\beta)}{(1-\beta)^3} = \lim_{\beta\to -1}(1+\beta)g(\beta) = 0\,,$$

it follows from Lemma A in Appendix that

$$S^\varepsilon(n) = \frac{1}{p}\sum_{k=1}^{p-1} g(\alpha^k) = -\frac{6}{p}(\lambda n-\varepsilon)^3 \qquad (\text{mod } \mathbb{Z}).$$

Hence it follows from the same calculation as in M_1 that

$$\widehat{f_\Omega}(\sigma) \equiv = -\frac{6}{4}\frac{192}{p}\lambda^2 = -\frac{288}{p}\lambda^2 \qquad (\text{mod } \mathbb{Z}).$$

Therefore $\widehat{f_\Omega}(\sigma) \neq 0$ if $p \geq 5$ and p is not a divisor of λ.

Corollary 2.2. *M_k does not admit any Kähler metric of constant scalar curvature whose Kähler form is contained in an integral multiple of $c_1(M_k)$ if $k = 1, 2$.*

Remark 2.3. *Using the results of Lichnerowicz* [7], [8], *we can also prove the corollary above by investigating the structure of the Lie algebra $H(M_k)$ $(k = 1, 2)$.*

3. Fixed point data of an action on manifolds of constant scalar curvature

Theorem 1.4 can be used as the restrictive condition on the fixed point data. Let $\mathbb{CP}^{d_j}$ $(1 \leq j \leq \ell)$ be complex projective spaces and $M = \mathbb{CP}^{d_1}\times\cdots\times\mathbb{CP}^{d_\ell}$ the direct product of $\mathbb{CP}^{d_j}$'s, which is a $\sum_j d_j$-dimensional complex manifold. Let g_j be the standard Kähler metric on $\mathbb{CP}^{d_j}$ and g the Kähler metric on M defined by the direct product of g_j's. Let $\mathrm{Isom}(M)$ denote the subgroup of $\mathrm{Aut}(M)$ consisting of isometries with respect to g. Then $\mathrm{Isom}(\mathbb{CP}^{d_j})$ contains the circle group S^1 for $1 \leq j \leq \ell$. Let $\pi_j : M \longrightarrow \mathbb{CP}^{d_j}$ be the j-th factor projection and H_j $(1 \leq j \leq \ell)$ the hyperplane bundle over $\mathbb{CP}^{d_j}$. Set $L = \otimes_{j=1}^{\ell}\pi_j^* H_j^{\lambda_j}$ for $\lambda_j \in \mathbb{Z}$ $(1 \leq j \leq \ell)$. Then any Hodge class Ω is equal to $c_1(L)$ for some integers λ_j.

Thorem 3.1. *The product action of $S^1 \subset Isom(\mathbb{CP}^{d_j}) \subset Isom(M)$ on M lifts to an action on L, and the fixed point data of the element $\sigma \in S^1 \subset Isom(M)$ of finite order satisfies the condition that $F_L(\sigma) \in \mathbb{Z} + \mu_\Omega \mathbb{Z}$ for any λ_j $(1 \le j \le \ell)$ and for any lifted S^1-action on L.*

Proof. Let σ_j be an cyclic element of $S^1 \subset \mathbb{CP}^{d_j}$. Then σ_j acts on H_j as follows:

$$\sigma_j \cdot [[z_0 : \cdots : z_{d_j}], h_j] = [\sigma_j \cdot [z_0 : \cdots : z_{d_j}], h_j] = [[w_0 : \cdots : w_{d_j}], h_j]$$

where h_j is an element of the fiber of H_j at $[z_0 : \cdots : z_{d_j}]$. Since any biholomorphic automorphism of $\mathbb{CP}^{d_j}$ is linear, we can show that the action defined above is well-defined as follows:

$$\begin{aligned}\sigma_j \cdot [[z_0 : \cdots : z_{d_j}], h_j] &= \sigma_j \cdot [[cz_0 : \cdots : cz_{d_j}], ch_j] = [\sigma \cdot [cz_0 : \cdots : cz_{d_j}], ch_j] \\ &= [[cw_0 : \cdots : cw_{d_j}], ch_j] = [[w_0 : \cdots : w_{d_j}], h_j]\end{aligned}$$

where c is a complex number. The product action $\sigma = \sigma_1 \times \cdots \times \sigma_j \in \mathrm{Isom}(M)$ clearly lifts to an action on L. Here since the scalar curvature of g is the sum of the scalar curvatures of g_j's and the scalar curvature of g_j is constant, g is a Kähler metric of constant scalar curvature. Therefore it follows from Theorem 1.1 and Theorem 1.4 that $F_L(\sigma) \in \mathbb{Z} + \mu_\Omega \mathbb{Z}$.

Example 3.2. *Let M be the 2-dimensional complex manifold defined by the direct product $M = \mathbb{CP}^1_1 \times \mathbb{CP}^1_2$ where both of $\mathbb{CP}^1_1$, $\mathbb{CP}^1_2$ are the 1-dimensional complex projective spaces $\mathbb{CP}^1$. Let H_i denote the hyperplane bundles over $\mathbb{CP}^1_i$ and $\pi_i : M \longrightarrow \mathbb{CP}^1_i$ the projection for $i = 1, 2$. Let L be the complex line bundle over M defined by $L = \pi_1^* H^\lambda \otimes \pi_2^* H^\mu$ for any integers λ, μ. Set $x = c_1(\pi_1^* H) \in H^2(M;\mathbb{Z})$, $y = c_1(\pi_2^* H) \in H^2(M;\mathbb{Z})$. Then any element of $H^2(M;\mathbb{Z})$ is expressed as a linear combination of x, y and any Hodge class Ω is equal to $c_1(L) = \lambda x + \mu y$ for some integers λ, μ. Then since $c_1(M) = 2x + 2y$ and $x^k y^\ell [M] = \delta_{k1}\delta_{\ell 1}$, we have*

$$\mu_\Omega = \frac{xy - \text{coeff. of } (\lambda x + \mu y)(2x + 2y)}{xy - \text{coeff. of } (\lambda x + \mu y)^2} = \frac{\lambda + \mu}{\lambda \mu}. \tag{1}$$

Let p be an odd prime number. Then the cyclic group $\mathbb{Z}_p = \langle \sigma \rangle$ acts on M by

$$\sigma \cdot ([z_0 : z_1], [u_0 : u_1]) = ([z_0 : z_1], [\alpha u_0 : u_1]) = ([z_0 : z_1], [u_0 : \alpha^{-1} u_1]).$$

This action naturally extends to an action of $S^1 \subset Isom(M)$ and lifts to an action on $\pi_i^* H$ $(i = 1, 2)$ as follows:

$$\begin{aligned}\sigma \cdot [([z_0 : z_1], [u_0 : u_1]), (h_1, h_2)] &= [([z_0 : z_1], [\alpha u_0 : u_1]), (h_1, h_2)] \\ &= [([z_0 : z_1], [u_0 : \alpha^{-1} u_1]), (h_1, \alpha^{-1} h_2)]\end{aligned}$$

where h_i is an element of the fiber of $\pi_i^* H$ at $([z_0 : z_1], [u_0 : u_1])$ for $i = 1, 2$. This action naturally defines an lifted action on L and this lifted action clearly extends to an S^1-action.

The fixed point set of the action consists of the following two connected components:

$$N_1 = \{([z_0 : z_1], [1 : 0])\} \simeq \mathbb{CP}^1 \times q_0, \quad N_2 = \{([z_0 : z_1], [0 : 1])\} \simeq \mathbb{CP}^1 \times q_1$$

where $q_0 = [1 : 0]$, $q_1 = [0 : 1]$ are points in $\mathbb{CP}^1$. It is clear that N_i $(i = 1, 2)$ have cell decompositions with no codimension one cells. Since it is obvious that $x = x|N_i$ is the positive generator of $H^2(N_i;\mathbb{Z})$ such that $x^k[N_i] = \delta_{k1}$, $y|N_i = 0$ and the total Chern class of TN_i is equal to $(1 + x)^2$, we have

$$\mathrm{Td}(TN_i) = \left(\frac{x}{1 - e^{-x}}\right)^2 \tag{2}$$

$$c_1(L|N_i) = \lambda x. \tag{3}$$

for $i = 1, 2$. Let $\nu(N_i, M)$ be the normal bundle of N_i in M. Then since $\nu(N_i, M)$ is the trivial bundle, it follows from the equality $c_1(K_M^{-1}|N_i) = c_1(TN_i) + c_1(\nu(N_i, M))$ that

$$c_1(K_M^\varepsilon|N_i) = -2\varepsilon x \tag{4}$$

for $i = 1, 2$. Moreover since

$$\begin{aligned}&\sigma \cdot ([z_0 : z_1], [1 : \tau]) = ([z_0 : z_1], [1 : \alpha^{-1}\tau]), \quad \sigma \cdot ([z_0 : z_1], [\tau : 1]) = ([z_0 : z_1], [\alpha\tau : 1]), \\ &\sigma \cdot [([z_0 : z_1], [1 : 0]), (h_1, h_2)] = [([z_0 : z_1], [\alpha : 0]), (h_1, h_2)] = [([z_0 : z_1], [1 : 0]), (h_1, \alpha^{-1} h_2)], \\ &\sigma \cdot [([z_0 : z_1], [0 : 1]), (h_1, h_2)] = [([z_0 : z_1], [0 : 1]), (h_1, h_2)],\end{aligned}$$

we have

$$\Phi(\nu(N_1, M)) = \frac{1}{1 - \alpha^k}, \quad \Phi(\nu(N_1, M)) = \frac{1}{1 - \alpha^{-k}}, \tag{5}$$

$$\sigma^k|(K_M^\varepsilon|N_1) = \alpha^{k\varepsilon}, \quad \sigma^k|(K_M^\varepsilon|N_2) = \alpha^{-k\varepsilon}. \tag{6}$$

$$\sigma^k|(L|N_1) = \alpha^{-k\mu}\,, \quad \sigma^k|(L|N_2) = \alpha^0\,. \tag{7}$$

Now it follows from Theorem 1.4 and the equalities (3), (4), (5), (6), (7), (8), (9) *that*

$$\begin{aligned}
S^{\varepsilon}(n) = x - &\text{coeff. of}\\
&\frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{1-\alpha^k}\left(\alpha^{k(\varepsilon-n\mu)}e^{(-2\varepsilon+n\lambda)x}-1\right)^3\left(\frac{x}{1-e^{-x}}\right)^2\frac{1}{1-\alpha^k}\\
&+\frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{1-\alpha^k}\left(\alpha^{k(-\varepsilon)}e^{(-2\varepsilon+n\lambda)x}-1\right)^3\left(\frac{x}{1-e^{-x}}\right)^2\frac{1}{1-\alpha^{-k}}\\
= x - &\text{coeff. of}\\
&\frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{(1-\alpha^k)^2}\left(\frac{x}{1-e^{-x}}\right)^2\left\{\left(\alpha^{k(\varepsilon-n\mu)}e^{(-2\varepsilon+n\lambda)x}-1\right)^3-\alpha^k\left(\alpha^{k(-\varepsilon)}e^{(-2\varepsilon+n\lambda)x}-1\right)^3\right\}\\
= x - &\text{coeff. of}\\
&\frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{(\alpha^k-1)^2}(1+x)\\
&\qquad\left\{\left(\alpha^{k(\varepsilon-n\mu)}(1+(-2\varepsilon+n\lambda)x)-1\right)^3-\alpha^k\left(\alpha^{k(-\varepsilon)}(1+(-2\varepsilon+n\lambda)x)-1\right)^3\right\}\\
= x - &\text{coeff. of}\\
&\frac{1}{p}\sum_{k=1}^{p-1}\frac{1}{(\alpha^k-1)^2}(1+x)\\
&\qquad\left\{\left(\alpha^{k(\varepsilon-n\mu)}-1\right)^3-\alpha^k\left(\alpha^{k(-\varepsilon)}-1\right)^3\right.\\
&\qquad\quad\left.+3\left(\alpha^{k(\varepsilon-n\mu)}\left(\alpha^{k(\varepsilon-n\mu)}-1\right)^2-\alpha^k\alpha^{k(-\varepsilon)}\left(\alpha^{k(-\varepsilon)}-1\right)^2\right)(-2\varepsilon+n\lambda)\,x\right\}\\
= \frac{1}{p}&\sum_{k=1}^{p-1}\left\{3\frac{\left(\alpha^{k(\varepsilon-n\mu)}\left(\alpha^{k(\varepsilon-n\mu)}-1\right)^2-\alpha^k\alpha^{k(-\varepsilon)}\left(\alpha^{k(-\varepsilon)}-1\right)^2\right)(-2\varepsilon+n\lambda)}{(\alpha^k-1)^2}\right.\\
&\qquad\left.+\frac{\left(\alpha^{k(\varepsilon-n\mu)}-1\right)^3-\alpha^k\left(\alpha^{k(-\varepsilon)}-1\right)^3}{(\alpha^k-1)^2}\right\}
\end{aligned}$$

Hence it follows from Lemma C in Appendix, we have

$$S^{\varepsilon}(n) = -\frac{1}{p}3\left((\varepsilon-n\mu)^2-(-\varepsilon)^2\right)(-2\varepsilon+n\lambda) = -\frac{1}{p}\left(3\lambda\mu^2n^3-6\varepsilon\mu(\lambda+\mu)n^2+12\mu n\right).$$

Hence it follows from Lemma B in Appendix that

$$\begin{aligned}
\sum_{j=1}^{2}(-1)^j\binom{2}{j}\left(S^{-1}(2-2j)-S^{+1}(2-2j)\right) &= -\frac{1}{p}12\mu(\lambda+\mu)\sum_{j=1}^{2}(-1)^j\binom{2}{j}(2-2j)^2\\
&= -\frac{1}{p}12\mu(\lambda+\mu)\cdot 2^2 2! = -\frac{96}{p}\mu(\lambda+\mu)\,,
\end{aligned}$$

$$\sum_{j=1}^{3}(-1)^j\binom{3}{j}S^0(3-2j) = -\frac{1}{p}3\lambda\mu^2\sum_{j=1}^{3}(-1)^j\binom{3}{j}n^3 = -\frac{1}{p}3\lambda\mu^2\cdot 2^3 3! = -\frac{144}{p}\lambda\mu^2\,.$$

Therefore it follows from (1) *that*

$$F_L(\sigma) = -\frac{1}{p}\left\{3\cdot 96\mu(\lambda+\mu) - 2\frac{\lambda+\mu}{\lambda\mu}144\lambda\mu^2\right\} = 0$$

for any λ, μ.

APPENDIX

Lamma A Assume that $(x+1)(x-1)^{\ell}g(x)$ is an integral polynomial $\Phi(x)$ for a natural number ℓ and that

$$\lim_{x\to 1}(x+1)g(x) = \lim_{x\to 1}\frac{\Phi(x)}{(x-1)^{\ell}} = \lambda\,, \quad \lim_{x\to -1}(x+1)g(x) = \lim_{x\to -1}\frac{\Phi(x)}{(x-1)^{\ell}} = \mu\,.$$

Then we have

$$\frac{1}{p}\sum_{k=1}^{p-1} g(\alpha^k) \equiv -\frac{\lambda}{2p} + \frac{\mu}{2} \pmod{\mathbb{Z}}.$$

Proof. Since the former equality implies that $\frac{\Phi(x)}{(x-1)^\ell}$ is an integral polynomial $P(x)$, there exists an integral polynomial $Q(x)$ such that

$$g(x) = \frac{\Phi(x)}{(x+1)(x-1)^\ell} = \frac{P(x)}{x+1} = Q(x) + \frac{\mu}{x+1}.$$

Here since $\sum_{k=1}^{p-1} \alpha^{mk} \equiv -1 \pmod p$ for any integer m, we have

$$\sum_{k=1}^{p-1} Q(\alpha^k) \equiv -Q(1) \pmod p,$$

and since

$$\mathrm{Re}\left(\frac{1}{\alpha^k+1}\right) = \mathrm{Re}\left(\frac{\overline{\alpha^k}+1}{|\alpha^k+1|^2}\right) = \frac{1+\cos\frac{2\pi}{p}}{2+2\cos\frac{2\pi}{p}} = \frac{1}{2},$$

we have

$$\sum_{k=1}^{p-1}\frac{1}{\alpha^k+1} = \sum_{k=1}^{p-1}\mathrm{Re}\left(\frac{1}{\alpha^k+1}\right) = \frac{p-1}{2}.$$

Therefore it follows that

$$\sum_{k=1}^{p-1} g(\alpha^k) = \sum_{k=1}^{p-1} Q(\alpha^k) + \sum_{k=1}^{p-1}\frac{\mu}{\alpha^k+1} \equiv -Q(1) + \frac{p-1}{2}\mu = -g(1) + \frac{\mu}{2} + \frac{p-1}{2}\mu$$

$$= -\lim_{x\to 1}\frac{\Phi(x)}{(x+1)(x-1)^\ell} + \frac{p\mu}{2} = -\frac{\lambda}{2} + \frac{p\mu}{2} \pmod p.$$

The equality of the lemma immediately follows from the equality above.

Lemma B For non-negative integers k, ℓ, the next equality holds.

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-2i)^\ell = \begin{cases} 0 & \text{if } \ell < k \text{ or } \ell = k+1 \\ 2^k k! & \text{if } \ell = k \end{cases}$$

Proof. Let k, ℓ be non-negative integers. First we prove the next equality:

(8) $$\sum_{i=0}^{k}(-1)^i\binom{k}{i} i^\ell = \begin{cases} 0 & (\ell < k) \\ (-1)^k k! & (\ell = k) \\ \frac{(-1)^k}{2}k(k+1)! & (\ell = k+1) \end{cases}.$$

Set $f(x) = (1+x)^k$. Then since $f^{(\ell)}(-1) = 0$ for $0 \le \ell < k$, it follows from the binomial theorem that

$$\sum_{i=0}^{k}(-1)^{i-\ell}\binom{\lambda}{i} i(i-1)\cdots(i-\ell+1) = 0 \iff \sum_{i=0}^{k}(-1)^i\binom{\lambda}{i} i(i-1)\cdots(i-\ell+1) = 0.$$

Using the equality above, we can prove the equality

(9) $$\sum_{i=0}^{k}(-1)^i\binom{k}{i} i^\ell = 0$$

for $0 \le \ell < k$ by induction. Next set

$$N(k) = \sum_{i=0}^{k}(-1)^i\binom{k}{i} i^k.$$

Then we have $N(1) = \sum_{i=0}^{1}(-1)^i\binom{1}{i} i = -1$ and it follows from (2) that

$$N(m+1) = \sum_{i=1}^{m+1}(-1)^i\binom{m+1}{i} i^{m+1} = -\sum_{i=0}^{m}(-1)^i\binom{m+1}{i+1}(i+1)^{m+1}$$

$$= -(m+1)\sum_{i=0}^{m}(-1)^i\binom{m}{i}(i+1)^m = -(m+1)\sum_{j=0}^{m}\binom{m}{j}\sum_{i=0}^{m}(-1)^i\binom{m}{i}i^j$$
$$= -(m+1)\sum_{i=0}^{m}(-1)^i\binom{m}{i}i^m = -(m+1)N(m)\,,$$

and hence it follows that $N(m) = (-1)^m m!$. Set

$$M(k) = \sum_{i=0}^{k}(-1)^i\binom{k}{i}i^{k+1}\,.$$

Then $M(1) = -1$ and it follows from (2) that

$$M(k) = \sum_{i=1}^{k}(-1)^i\frac{k!}{(k-i)!i!}i^{k+1} = k\sum_{i=1}^{k}(-1)^i\frac{(k-1)!}{(k-i)!(i-1)!}i^k = k\sum_{i=1}^{k}(-1)^i\binom{k-1}{i-1}i^k$$
$$= -k\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}(j+1)^k = -k\left(\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}j^k + k\sum_{j=0}^{k-1}(-1)^j\binom{k-1}{j}j^{k-1}\right)$$
$$= -kM(k-1) - k^2(-1)^{k-1}(k-1)! = -kM(k-1) + (-1)^k k\,k!$$
$$\Longleftrightarrow\ a_k = a_{k-1} + k \quad \text{where} \quad a_k = \frac{M(k)}{(-1)^k k!}$$
$$\Longleftrightarrow\ a_k = k + k - 1 + \cdots + 1 = \frac{k(k+1)}{2}$$
$$\Longleftrightarrow\ M(k) = (-1)^k k!\frac{k(k+1)}{2} = \frac{(-1)^k}{2}k(k+1)!\,.$$

This completes the proof of (8). Note that it easily follows from (8) that

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-2i)^\ell = \sum_{j=0}^{\ell}\binom{\ell}{j}k^j\sum_{i=0}^{k}(-1)^i\binom{k}{i}(-2i)^{\ell-j} = 0 \quad \text{if } \ell < k\,.$$

Moreover using (8), we have

$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-2i)^k = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(-2i)^k = (-2)^k N(k) = 2^k k!$$
$$\sum_{i=0}^{k}(-1)^i\binom{k}{i}(k-2i)^{k+1} = \sum_{i=0}^{k}(-1)^i\binom{k}{i}(-2i)^{k+1} + (k+1)k\sum_{i=0}^{k}(-1)^i\binom{k}{i}(-2i)^k$$
$$= (-2)^{k+1}\frac{(-1)^k}{2}k(k+1)! + (k+1)k(-2)^k(-1)^k k! = -2^k k(k+1)! + 2^k k(k+1)! = 0$$

Lemma C For any integers ξ, η, the next equalities holds.

$$\frac{1}{p}\sum_{k=1}^{p-1}\frac{\alpha^{k\xi}(\alpha^{k\eta}-1)^2}{(\alpha^k-1)^2} \equiv -\frac{1}{p}\eta^2 \pmod{\mathbb{Z}}\,, \quad \frac{1}{p}\sum_{k=1}^{p-1}\frac{\alpha^{k\xi}(\alpha^{k\eta}-1)^3}{(\alpha^k-1)^2} \equiv 0 \pmod{\mathbb{Z}}$$

Proof. Set

$$g(x) = \frac{x^\xi(x^\eta-1)^2}{(x-1)^2}\,, \quad h(x) = \frac{x^\xi(x^\eta-1)^3}{(x-1)^2}\,.$$

Then since

$$\lim_{x\to 1}(x+1)g(x) = 2\eta^2\,,\ \lim_{x\to -1}(x+1)g(x) = 0\,,\ \lim_{x\to 1}h(x) = 0\,,\ \lim_{x\to -1}(x+1)h(x) = 0\,,$$

it follows from Lemma A in Appendix that

$$\frac{1}{p}\sum_{k=1}^{p-1}g(\alpha^k) \equiv -\frac{1}{p}\eta^2 \pmod{\mathbb{Z}}\,, \quad \frac{1}{p}\sum_{k=1}^{p-1}h(\alpha^k) \equiv 0 \pmod{\mathbb{Z}}\,.$$

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定スカラー曲率ケーラー計量を持つ複素多様体の BCF 特性数について

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この論文において，BCF 特性数を用いることによって，指定されたホッジ類に対しそのホッジ類に含まれるケーラー類に対応する定スカラー曲率ケーラー計量を許容しない複素多様体の例を与える．また，定スカラー曲率ケーラー計量を許容する複素多様体が有限位数の自己同型写像を持つとき，その自己同型写像の固定点集合に対する制約的条件を与える．

キーワード: BCF 特性数，複素多様体，ケーラー計量，定スカラー曲率，固定点集合