Measuring Chaos: Topological Entropy and Correlation Dimension in Discrete Maps

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Abstract

Nonlinear systems may exhibit chaos during evolution and at the state of chaos one sees the emergence of certain chaotic attractors. Chaotic set appears during the phenomena of bifurcations while varying certain parameter. The Lyapunov characteristic exponents (LCE) and topological entropies are both suitable tools for description of the transition to chaos. Plots of both of these indicators are similar as for identification of chaos. But LCE has limitation that it would not work for systems having relativistic considerations. However, the topological entropy is considered as a nice way to measure the complexity of a system such as chaos in the sense that the more complexity in the system means more topological entropy it will have.

In the present work, appearance of chaos through bifurcation in some one dimensional discrete nonlinear systems have been considered and plots of Lyapunov exponents and topological entropy for such evolution have been obtained. Then, the calculation of correlation dimensions of such chaotic sets, chaotic attractors, have been carried out. Graphical results reveal some interesting informations.

Key words: Topological entropy, Correlation dimension, Lyapunov exponents, Chaotic attractors

1 Introduction

Lyapunov characteristic exponents, (LCE), are very effective tools for identification of regular and chaotic motions sine these measure the degree of sensitivity to initial conditions in a system. If the divergence is exponential in time, with the constant factor in the exponent λ, then λ is a LCE of the system and if $\lambda > 0$, then the system be chaotic. The system is regular as long as $\lambda \leqq 0$. However, the usefulness of Lyapunov exponents are limited to non-relativistic systems only, Gribble (1995). An another good way to identify the complexity or chaotic nature of a dynamical system can be represented through its numerical measure of topological entropy: the more complex a system is, the more topological entropy it will have. Topological entropy was introduced by Adler et al (1965) and later extended by Bowen (1971). After these works a number of research articles appeared on different systems and computation methodology of topological entropy by Boyarski and Gora (1991), Balmforth et al(1994), Stewart and Edward (1997) , Iwai (1998) and many others. A recent survey on topological entropy can be obtained in the article by Kawan (2011).

A chaotic attractor is composed of a complex pattern. For a wide variety of systems evolving with time, an alternate replacement of Lyapunov exponents which could be more reliable and acceptable as indicator is the topological entropy. Topological entropy describes the rate of mixing of a dynamical system. It has a relationship to both LCE, through the dependence of rate, and to the ergodicity, because of the association of mixing. Further, the topological entropy can be thought of an information on the theoretic property of a system. If we measure the state of a system to a finite degree of accuracy, then topological entropy measures the rate at which we can gain more information about the system as it evolve over time. Topological entropy is global in nature and provides a true measure of chaoticity of a system. For a system having non-zero topological entropy, the rate of mixing must be exponential which is reminiscent of LCEs. But such exponentiality of mixing is not relative to time, but rather to the number of discrete steps through which the system has evolved. Topological entropy gives measure of the complexity of systems and provides a new theory of understanding about chaotic systems in a more realistic way, Mitchell (2009). Positivity of LCEs and topological entropy are characteristic of chaos in the system. A mathematical definition of topological entropy can be found in the book by Nagashima and Baba, (2005). Objective of the present work is to study evolutions in a number of one and two dimensional discrete maps. Main interest here is to observe the chaotic and non-chaotic nature of motions through bifurcations. For each map we have obtained the LCEs and correlation dimension of the chaotic set appearing in bifurcation. Also, we have obtained the topological entropy for each one dimensional systems and made comparison of graphs of LCEs and topological entropies.

2 Descriptions of Lyapunov exponents, Topological Entropies and Correlation Dimensions :

(a) Lyapunov Exponents:

The Lyapunov exponent, (or Lyapunov characteristic exponent LCE), provides an average measure of exponential divergence of two orbits initiated with infinitesimal separation. The largest eigenvalue of a complex dynamical system is an indicator of chaos, Saha and Budhraja (2007). For, consider two orbits initiated at x_0 and y_0 with x_0, $y_0 \in [0, 1]$, of a one dimension map

$$f : [0, 1] \to [0, 1]$$

such that $|x_0 - y_0| << 1$, then assuming $|x_n - y_n| << 1$, where $x_n = f^n(x_0), y_n = f^n(y_0)$ are respectively the nth iterations of x_0 and y_0 under f, by Taylor's theorem, one can obtains

$$|x_n - y_n| = \Big[\prod_{t=0}^{n-1} |f'(t)||x_0 - y_0|\Big]. \qquad (1)$$

Then, the exponential separation rate $\log |f'(x)|$ of two nearby initial conditions, averaged over the entire trajectory, can be given by

$$\lambda(x_0) = \lim_{n\to\infty} \frac{1}{n} \log\Big[\prod_{t=0}^{n-1} |f'(x_t)|\Big], \qquad (2)$$

where

$$\prod_{t=0}^{n-1} |f'(x_t)| \approx e^{\lambda(x_0)n}, \quad for \;\; n >> 1$$

and this implies

$$|x_n - y_n| \approx e^{\lambda(x_0)n}|x_0 - y_0|. \qquad (3)$$

For higher dimensional system, we can generalize the above one dimensional case to higher dimension and obtain

$$\lambda(X_0, U_0) = \lim_{n\to\infty} \frac{1}{n} \log \| \prod_{t=0}^{n-1} J(X_t)U_0\|, \qquad (4)$$

and

$$\|X_n - Y_n\| \approx e^{\lambda(x_0,U_0)n},$$

where $X \in \Re^n, F : \Re^n \to \Re^n, U_0 = X_0 - Y_0$ and J is the Jacobian matrix of map F. Quantitatively, two trajectories in phase space with initial separation δx_0diverge (provided that the divergence (can be treated within the linearized approximation)

$$|\delta x(t)| \approx e^{\lambda t}|\delta x(0)| \tag{5}$$

where $\lambda > 0$ is the Lyapunov exponent. The system described by the map f be regular as long as $\lambda \leqq 0$ and chaotic when $\lambda > 0$.

(b) Topological Entropy:

Topological entropy $h(f)$ for a map f defined in a closed interval $I = [a, b]$, is closely related to Li and Yorke chaos, Nagashima and Baba (2005), and measures the complexity of the map f.

If f be a continuous map from I to I and if α be an open initial cover of I, then the topological entropy $h(f)$ can be described by the supremum, $sup\ \ h(\alpha, f)$, for all the covers of interval I such that

$$h(\alpha, f) = \lim_{n\to\infty} \frac{1}{n} \log N\Big[\bigvee_{i=0}^{n-1} f^{-1}\alpha\Big], \tag{6}$$

then

$$h(f) = sup\ h(\alpha, f). \tag{7}$$

When the map f is piecewise-monotonic over I, the topological entropy can be determined by the lap number, $lap(f^n)$ of the iterated map f^n as follows :

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log lap(f^n), \tag{8}$$

The lap number of f grows with n in general. If the growth obeys the power law,

$$lap(f^n) \sim kn^{\alpha},$$

then by (8),

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log(kn^{\alpha}) = \lim_{n\to\infty} \frac{\alpha}{n} \log n = 0. \tag{9}$$

However, if it grow exponentially, $lap(f^n) \sim k\alpha^n, (\alpha > 1)$, then

$$h(f) = \lim_{n\to\infty} \frac{1}{n} \log(k\alpha^n) = \log \alpha. \tag{10}$$

This shows that $h(f)$ is determined by the way $lap(f^n)$ increases.

In case of superstable periodic orbits, the method of structure matrix can be employed. For take the case of logistic map $f(x) = \mu x(1 - x)$ when $\mu = 3.960270...$, (Nagashima and Baba, page 131, 2005), one can obtain the structure matrix $\mathbf{M}$ and then find out the largest eigenvalue, λ_{max} of $\mathbf{M}$. Then, the topological entropy can be obtained as

$$h = \log(\lambda_{max}). \tag{11}$$

(c) Correlation Dimensions:

As stated, chaos may exist in nonlinear systems during evolution and that can be seen easily by observing the bifurcation diagrams. A chaotic set, an strange attractor, has fractal structure. Correlation dimension gives a measure of dimensionality of the chaotic set. Being one of the characteristic invariants of nonlinear system dynamics, the correlation dimension actually gives a measure of complexity for the underlying attractor of the system. To determine correlation dimension we use statistical method. It is a very practical and efficient method then other methods, like box counting etc. The procedure to obtain correlation dimension follows the following steps, Martelli (1999):

Consider an orbit $O(\mathbf{x_1}) = \{\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}, \mathbf{x_4}, \}$, of a map $f : U \to U$, where U is an open bounded set in $\Re^n$. To compute correlation dimension of $O(\mathbf{x_1})$, for a given positive real number r, we form the correlation integral, Grassberger and Procaccia (1983),

$$C(r) = \lim_{n\to\infty} \frac{1}{n(n-1)} \sum_{i\neq j}^{n} H\Big[r - \parallel \mathbf{x_i} - \mathbf{x_j} \parallel\Big], \tag{12}$$

where

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geqq 0 \end{cases}$$

is the unit-step function, (Heaviside function). The summation indicates the number of pairs of vectors closer to r when $1 \leqq i, j \leqq n$ and $i \neq j$. $C(r)$ measures the density of pair of distinct vectors $\mathbf{x_i}$ and $\mathbf{x_j}$ that are closer to r. The correlation dimension D_c of $O(\mathbf{x_1})$ is defined as

$$D_c = \lim_{r\to 0} \frac{\log C(r)}{\log r} \tag{13}$$

To obtain D_c, $\log C(r)$ is plotted against $\log r$ and then we find a straight line fitted to this curve. The y-intercept of this straight line provides the value of the correlation dimension D_c.

3 Numerical Computations for Discrete Models:

In this section we have considered following one and two dimensional discrete maps, each representing some real phenomena:

3.1 One Dimensional Maps:

(a) Logistic Map:

$$f(x) = \mu x(1 - x) \tag{14}$$

This map appeared in numerous research articles, and so we do not intend here a detailed description for this map. The bifurcation diagram of this map is given in Fig.1 which clearly indicates how the system evolve chaotically after number of regular steps.

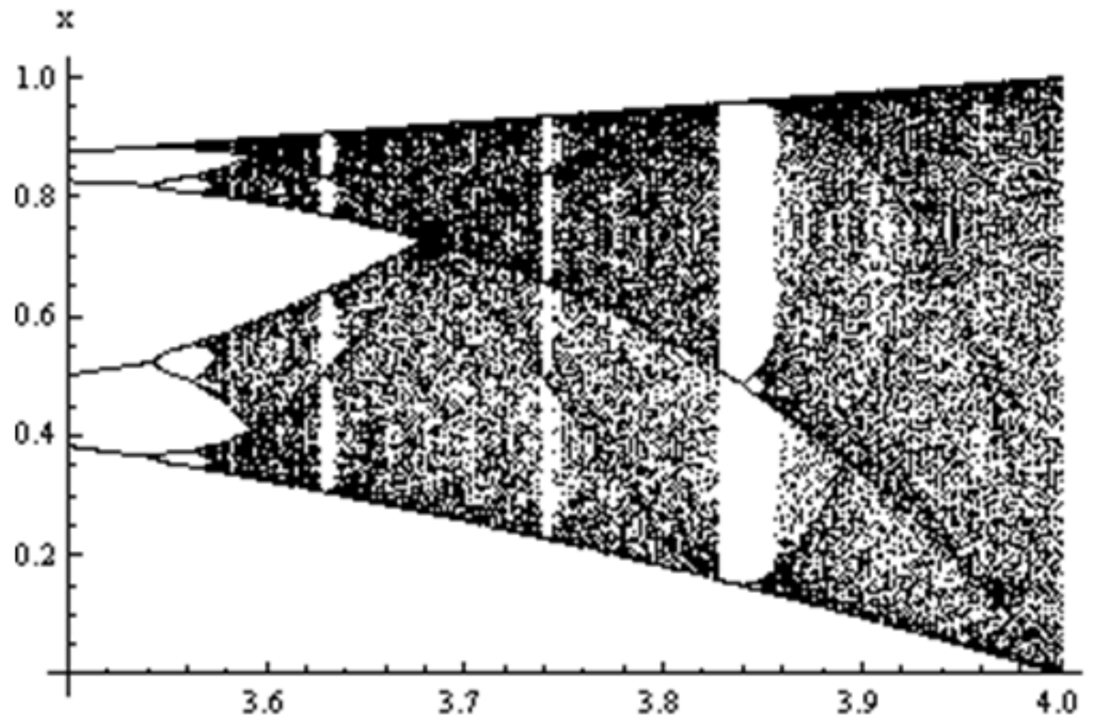


Fig.1: Bifurcation diagram of logistic map for $3.5 \leqq \mu \leqq 4$

The plots of LCEs and that for the topological entropy are shown in Fig. 2. We have also made a comparison of these two in Fig.2 (c).

The appearance of periodic windows within chaotic region, shown in bifurcation diagram in Fig.1, resulting in corresponding decrease in topological entropy and Lyapunov exponents shown, Fig.2(c).

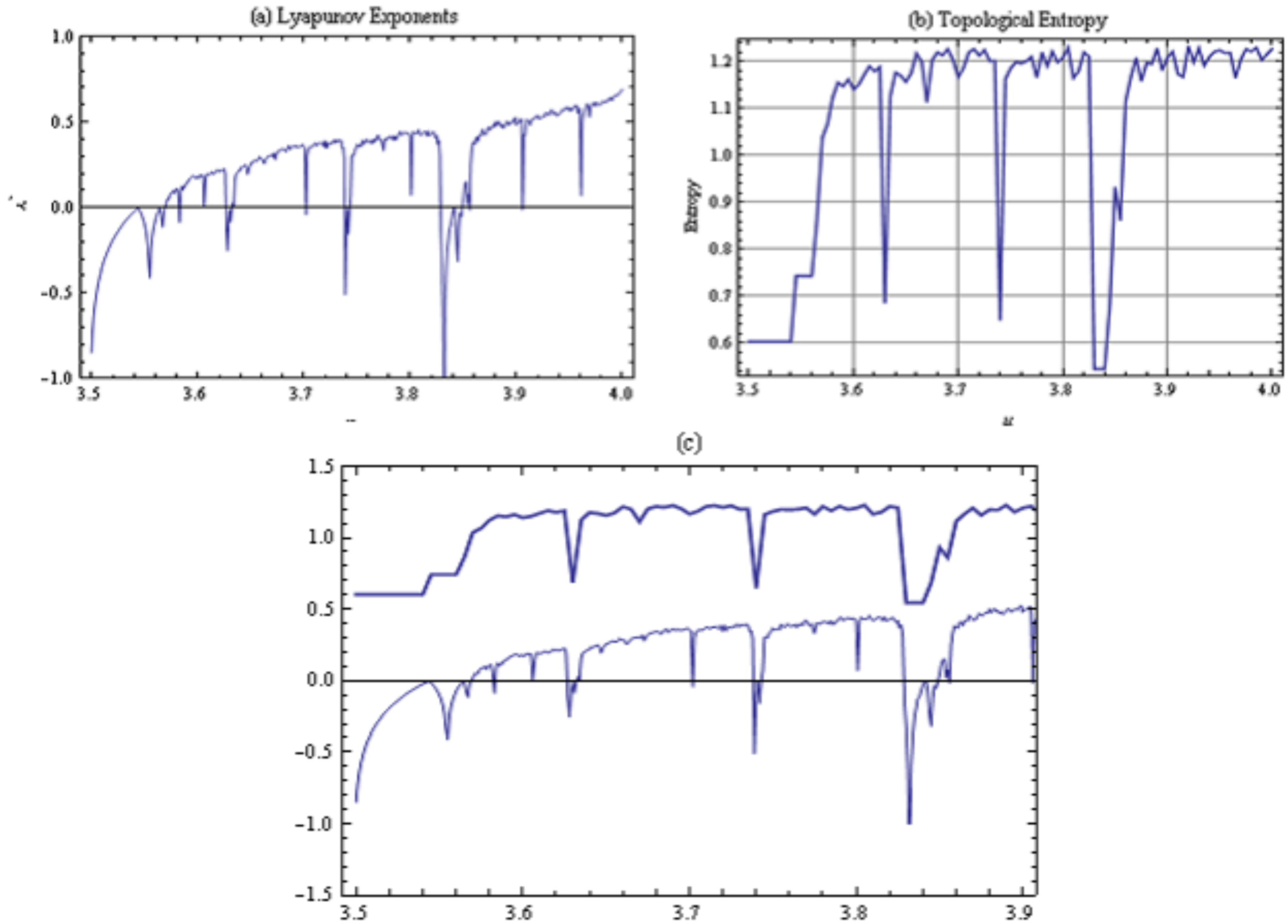


Fig. 2: Plots of logistic map: (a) Lyapunov exponents, (b) Topological entropy and (c) Comparison of these two plots. The upper curve in (c) plotted by adding 0.3 to the entropy values to prevent overlapping of curves.

(b) Modified Ricker Model for insect population:

First we consider the modified Ricker-type map described by following equation, Henson et al. (1999),

$$x_{n+1} = f_\alpha(t, x_n) = b[1 + \alpha(-1)^n]x_n e^{-cx_n} + (1-\mu)x_n \qquad (15)$$

Here, x_{n+1} represents the density of individuals in a population at census $n+1$ for a given density of individuals x_n at census n. The parameter $b > 0$ stands for the inherent per capita recruitment rate per census interval at small population sizes and e^{-cx_n} represents the fractional reduction of recruitment due to density- dependent effects. The parameter $\mu, 0 \leqq \mu \leqq 1$, represents the fraction of individuals expected to die during one census period. Then a certain period-forcing is introduced such that the birth rate oscillates with relative amplitude $0 < \alpha < 1$ and average b. Taking n even and odd positive integers and varying parameter b the bifurcation diagrams of (15) be represented respectively as, Fig. 3 (a) and 3(b).

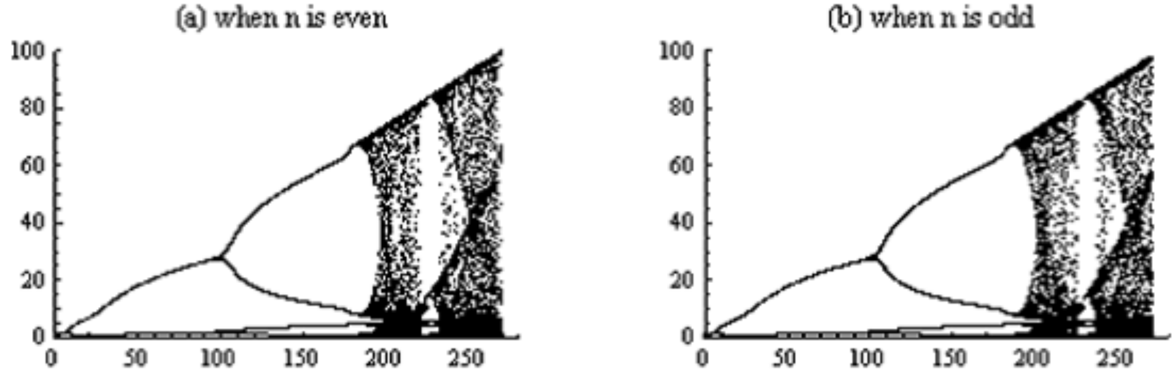


Fig. 3: Bifurcation of model described by eqn. (15) (a) when n is even and (b) when n is odd. Here, $\alpha = 0.01, c = 1.0, \mu = 0.93$ and b is varied $0 \leqq b \leqq 280$.

The corresponding plots for LCEs for above two cases are given in Fig. 4 (a) and Fig. 4(b) shown below.

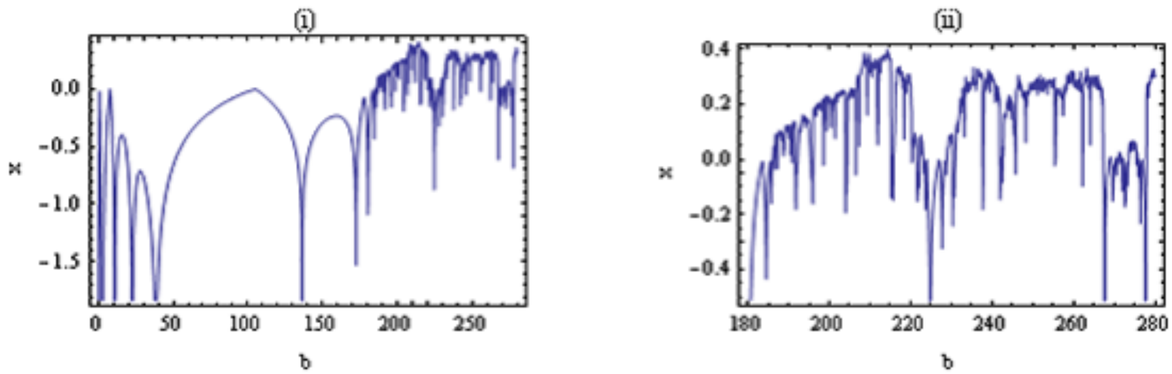


Fig. 4(a): LCE curve of equation (15) when n is even: the plot on the right is for the region of bifurcation diagram where periodic windows are seen. The parameter values are $c = 1.0, \mu = 0.93, \alpha = 0.01$, in (i) $0 \leqq b \leqq 280$ and in (ii) $180 \leqq b \leqq 280$.

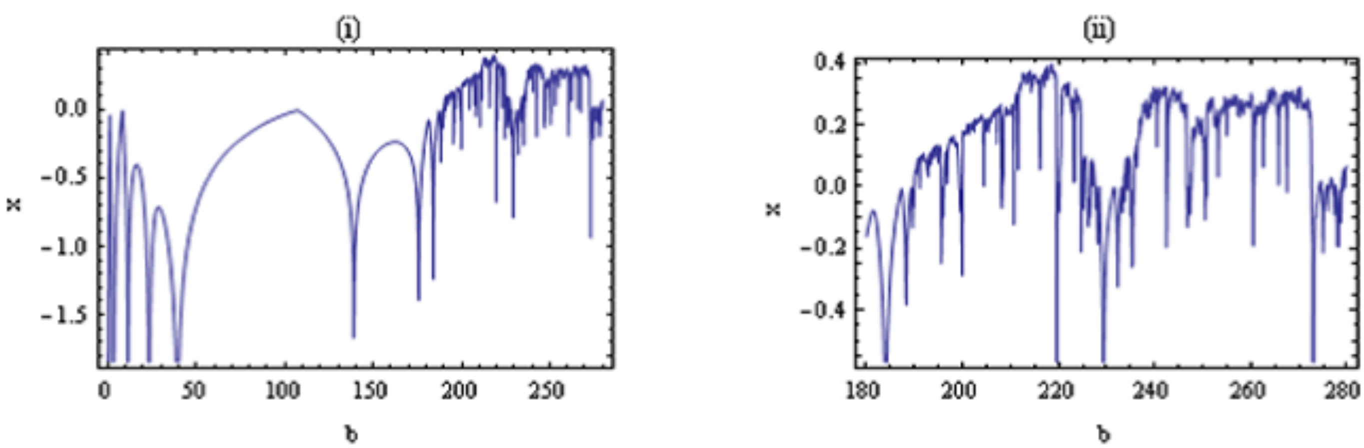
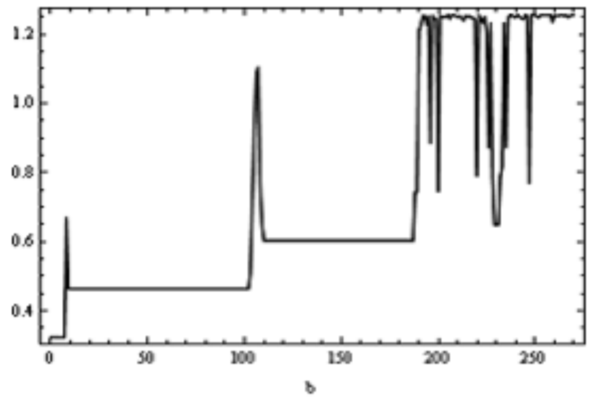


Fig. 4(b): LCE curve of equation (15) when n is odd: with values $c = 1.0, \mu = 0.93, \alpha = 0.01$;and in (i) $0 \leqq b \leqq 280$ & in (ii) $180 \leqq b \leqq 280$.The right hand figure is for the periodic window region.

The topological entropy for both cases are calculated numerically and represented, for both cases; n even positive integer and odd positive integer, by Fig. 5 (a) and 5 (b).



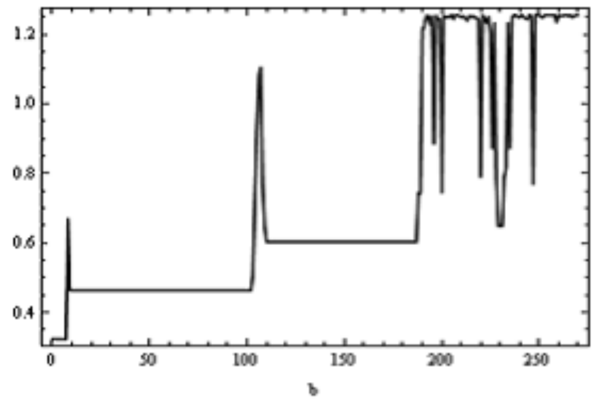


Fig. 5: Topological entropy of map (15) (a) when n is even and (b) when n is odd. Here, $\alpha = 0.01, c = 1.0, \mu = 0.93$ and b is varied $0 \leqq b \leqq 280$.

Comparison between LCEs plot and that for topological entropies can be made as in the earlier case of logistic map.

(c) Gaussian Map:

Another type of nonlinear one-dimensional iterative map is the Gaussian map $G : IR \to IR$ defined by

$$\begin{aligned} f(x) &= e^{-ax^2} + b, \\ or \quad x_{n+1} &= exp[-ax^2] + b \end{aligned} \tag{16}$$

where a and b are constants. The parameters a and b are related to the width and height of the Gaussian curve, respectively. Since there are two parameters associated with this map, one would expect the dynamics to be more complicated than those for the logistic map. All of the features which appear in the logistic map are also present for the Gaussian map. Also, some additional properties described as period dubblings, period undoublings, and bistability observed in Gauss map.. These features can be seen in the bifurcation diagrams. The map (16) has two points of inflection at $x = \pm\frac{1}{\sqrt{2a}}$. This means period-one behavior can exist for two ranges of the parameters that can make a transition from being stable to unstable and back to stable again. The bifurcation diagram of (16), by Fig. 6 presents meaningful explanation of evolutionary behavior.

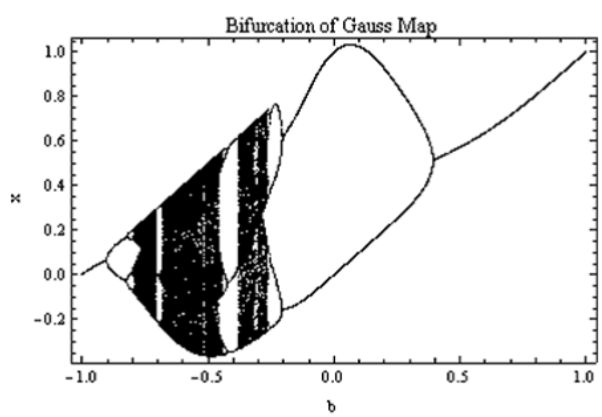


Fig. 6: Bifurcation diagram of Gauss map for $a = 8$ and $-1 \leqq b \leqq 1$.

The plots of LCEs and that for the topological entropy of the Gauss Map are shown in Fig. 7.

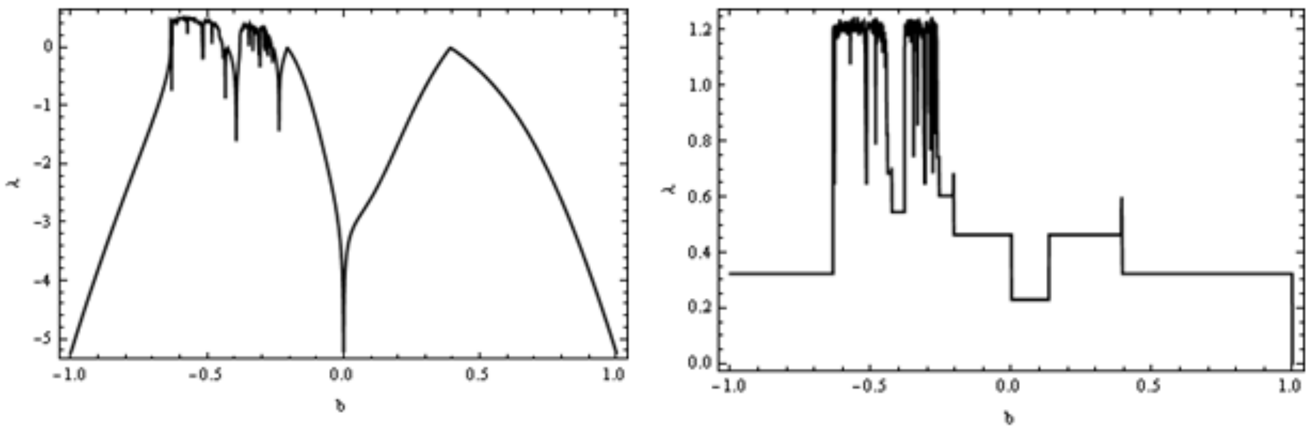


Fig. 7: LCEs and entropy diagrams of Gauss map for $a = 8$ and $-1 \leqq b \leqq 1$.

3.2 Two Dimensional Maps:

(a) Behrens - Feichtinger Model:

A formal description of Behrens-Feichtinger map, (B F model), is referred to the work by Hołyst et al (1996) and Matjaž Perc (2007). The discrete model is represented by map,

$$x_{n+1} = \frac{a}{1 + e^{-c(x_n - y_n)}} + (1 - \alpha)x_n$$
$$y_{n+1} = \frac{b}{1 + e^{-c(x_n - y_n)}} + (1 - \beta)y_n \quad (17)$$

For parameter values $\alpha = 0.46, \beta = 0.7, a = 0.16, b = 0.9, c = 105$, the map (17) evolve chaotically and that can be observed through the bifurcation diagram, Fig. 8.

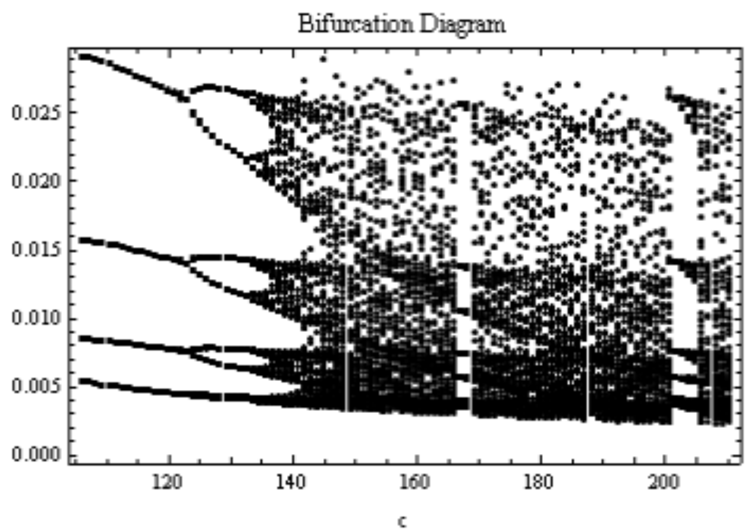


Fig. 8: Bifurcation diagram of B F model for the parameter $\alpha = 0.46, \beta = 0.7, a = 0.16, b = 0.9, c = 105$.

The diagram for LCEs is given in Fig. 9, which shows the chaotic motion of the system.

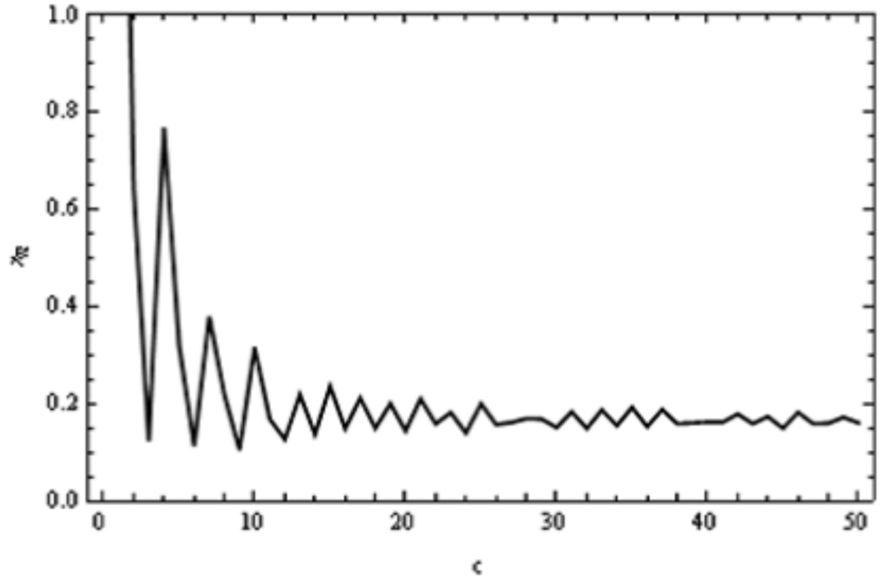


Fig. 9: LCEs diagram of B F model for the parameter $\alpha = 0.46, \beta = 0.7, a = 0.16, b = 0.9, c = 105$.

(b) The Bouncing Ball model:
A discrete form of equation of motion for bouncing ball is represented by

$$\begin{aligned} x_{n+1} &= x_n + a y_n \\ y_{n+1} &= k y_n + (1+k)\cos(x_n + a y_n) \end{aligned} \quad (18)$$

Its dynamic is very interesting and it has been discussed recently by Kamath et al (2008), Litak et al (2009) and Yuasa and Saha (2008). The bifurcation diagram and LCEs of model (3.5) is given below, left and right plot, respectively, of Fig. 10.

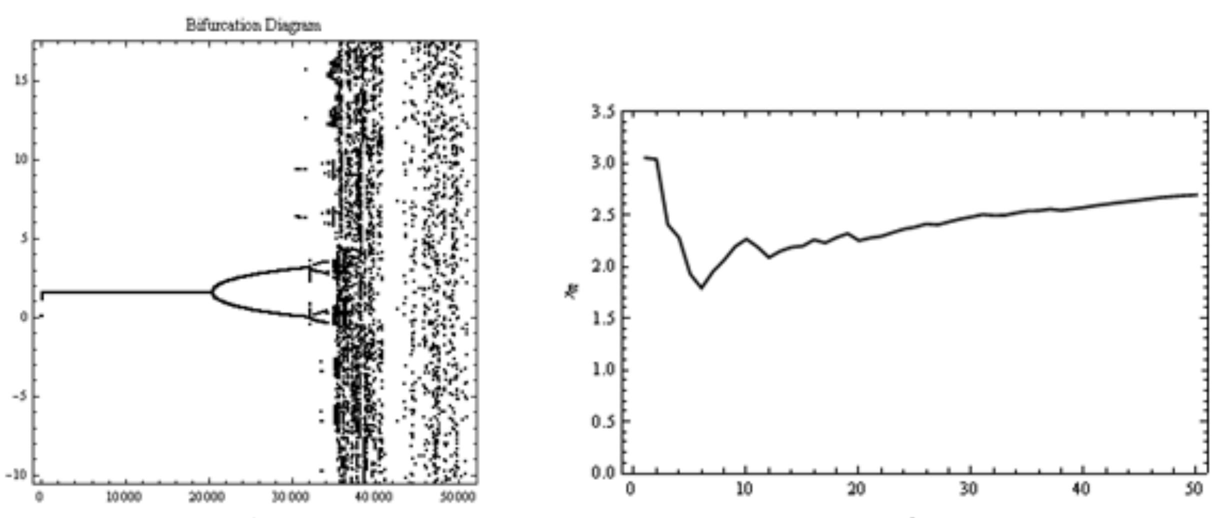


Fig. 10: Bifurcation diagram and LCEs plot of bouncing ball model.

4 Correlation Dimensions:

There can be various discrete models but, here we have selected only these few models. Our objective is to see the change in behavioral dynamics of the systems with regard to the change of control parameters by observing their bifurcation phenomena. The chaotic set appearing in bifurcation diagrams have fractal structure and so, it is necessary to compute dimension of those chaotic sets, (strange attractors). Thus we have calculated the correlation dimensions of the chaotic sets discussed in this work. For this we have collected data for the correlation curves plotted for each case and then used the method of least square linear fit described by Martelli (1999) and Nagashima and Baba (2005) and obtained correlation dimensions for each chaotic maps. Actually, the correlation dimension gives the measure of complexity whenever a system evolves chaotically.
Correlation curve and straight lines equations for models (14) to (18) are given below:

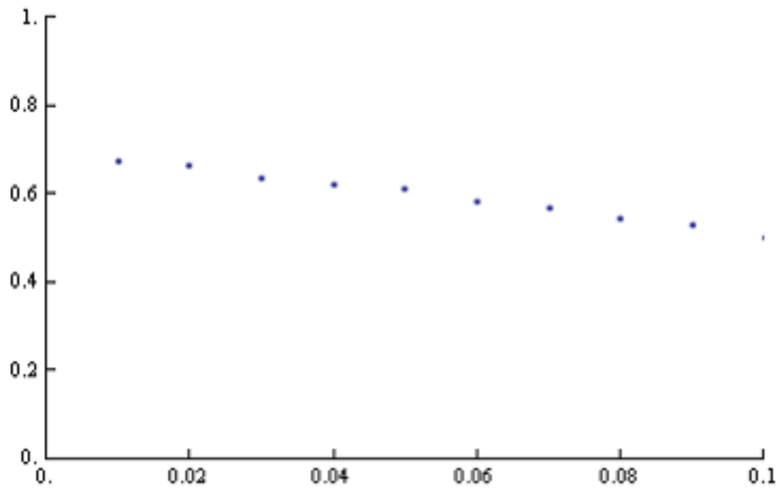


Fig. 11: Correlation curve of Logistic Map.

After drawing the correlation curve we have used least square linear fit to the data of correlation curves. This provides the equation of a straight line. For, logistic model, the straight line obtained is

$$y = 0.696758 - 1.89675x.$$

The y-intercept of this straight line is $0.696758 \approx 0.697$. So, Martelli (1999), this is the correlation dimension of the chaotic set of the logistic map. For two dimensional cases, sometimes, we have taken the slope of the straight line obtained by least square linear fit to the correlation data, Nagashima and Baba (2005). Below we have shown the correlation curves, Fig. 12 - Fig.14, and for each maps discussed in this article and used linear fit as stated above.

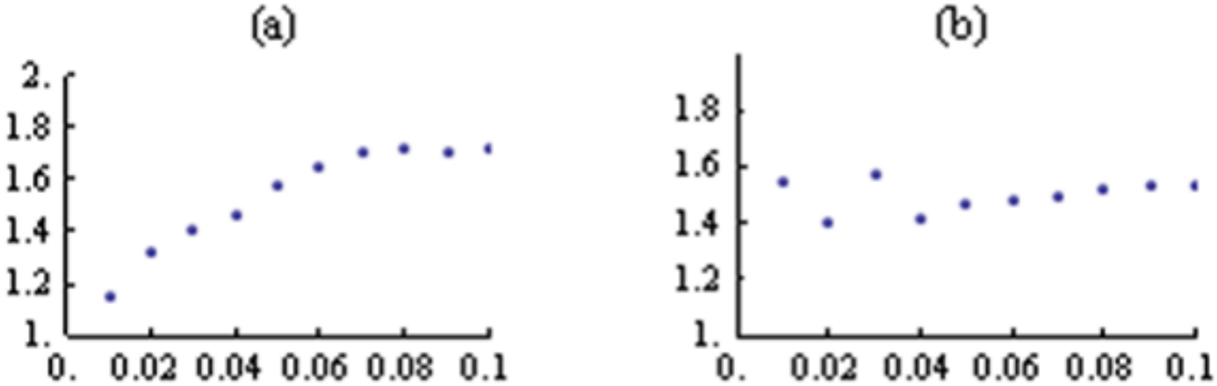


Fig. 12: Correlation curve of Map (15) (a) when n is even and (b) when n is odd.

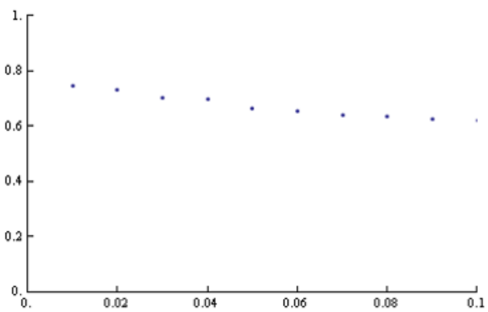


Fig. 13: Correlation curve of Gauss Map (16).

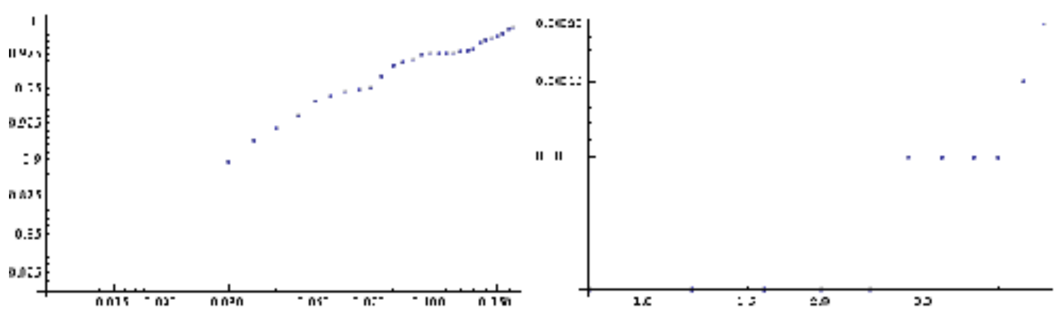


Fig. 14: Correlation curve of B F model (17) and that of Bouncing Ball model (18).

The correlation dimensions obtained are given in the table below.

S. No.	Map	Correlation Dimension
1	Logistic Map	0.696758
2	Modified Ricker Map (1) when n is even (2) when n is odd	(1) 1.20029 (2) 1.47021
3	Gauss Map	0.751797
4	Behrens-Feichtinger Model	0.304018
5	Bouncing Ball Model	0.747568

5 Conclusion

Specific models are chosen in this article to represent various dynamical systems. These models have large applications in different areas. By drawing bifurcation diagrams it was possible to detect the regular and chaotic regions of the system and to see the appearance of periodic orbits within chaos. LCE curves for each system justify such observations. Topological entropies are calculated for one dimensional models and shown they behave similar to those of LCEs. Correlation dimensions for each chaotic attractors have been obtained as these attractors are of fractal nature i.e. self similarity. Certain Mathematica codes are used in numerical calculations and plotting graphs. Computing topological entropies for two and higher dimensional systems will be in our future plan. The results obtained through this work are of important nature as the models used are from realistic considerations

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