An affirmative answer to a question concerning dense metrizable subspaces of generalized ordered spaces

Masami HOSOBUCHI

Abstract

In this paper, we give an answer to the following question that was posed by the author in [H3]: Let $(X, \mathcal{I})$ be a linearly ordered space with a dense metrizable subspace. Then, does the associated Sorgenfrey space $(X, \mathcal{S})$ have a dense metrizable subspace? Furthermore, we show some related consequences concerning σ-closed discrete dense subsets, and Property III that was defined in [BL].

Key words : Linearly ordered space, associated Sorgenfrey space, generalized ordered (GO) space, σ-closed discrete dense subset, dense metrizable subspace, Properties I, II, and III, G_δ-diagonal.

1. Introduction

Let $(X, <)$ be a linearly ordered set. We will consider two topologies on $(X, <)$ at the same time. One of them is a linearly ordered (topological) space (LOTS) and the other one is a Sorgenfrey space. Such a Sorgenfrey space is called the *associated* Sorgenfrey space in the connection with a given LOTS. A linearly ordered space $(X, <, \mathcal{I})$ has the order topology $\mathcal{I}$ defined by $<$, that is, a basic open neighborhood of z in the LOTS is of the form $]x, y[=\{u \in X : x < u < y\}$, where $x < z < y$ are points of X. The order topology is often called the interval topology. That is the reason the letter $\mathcal{I}$ is used. A basic open neighborhood of x in a Sorgenfrey space $(X, <, \mathcal{S})$ is of the form $[x, y[=\{z \in X : x \leq z < y\}$, where $x < y$. We usually abbreviate $(X, <, \mathcal{I})$ as $(X, \mathcal{I})$, and $(X, <, \mathcal{S})$ as $(X, \mathcal{S})$. We also write $(X, <)$ as X.

2. Linearly ordered spaces and Sorgenfrey spaces

H. R. Bennett, D. J. Lutzer and S. D. Purisch [BLP] defined the following properties to study dense subspaces of generalized ordered spaces. (See also [H2]). They are interesting and important because we have a fact : The density of X is equal to the cellularity of X.

A GO-space is defined as a subspace of a linearly ordered topological space that is abbreviated by LOTS. (See Section 1).

Definition 1. A (topological) space X is said to have *Property I* if and only if there exists a σ-closed discrete dense subset D of X, that is, $D = \cup\{D(n) : n \in N\}$ is a dense subset of X such that $D(n)$ is a closed discrete subset of X for every $n \in N$. N denotes the set of natural numbers.

Definition 2. A space X is said to have *Property II* if and only if there is a dense metrizable subspace of X.

Definition 3. A space X is said to have *Property III* if and only if, for each $n \in N$, there are an open subset $U(n)$ of X and a relatively closed discrete subset $D(n)$ of $U(n)$ such that, for a point p and an open subset G of X that contains p, there exists an $n \in N$ such that $p \in U(n)$ and $G \cap D(n) \neq \emptyset$. (See [BL], [H1]).

It is interesting to consider the relationships between $(X, \mathcal{T})$ and $(X, \mathcal{S})$, where X is a linearly ordered set. In this paper, we show that if $(X, \mathcal{T})$ has Property II, then so does $(X, \mathcal{S})$. (See Section 4). This answers the question that was asked by the author in [H3]. Furthermore, we investigate the cases of Properties I and III. (See Sections 3 and 5).

3. Conditions that assure Property I on the associated Sorgenfrey spaces

We showed in [H2] that a LOTS $(X, \mathcal{T})$ with Property I does not necessarily imply that the associated Sorgenfrey space $(X, \mathcal{S})$ has the property. For example, the double line $R \times \{0, 1\}$ with the usual lexicographic order-topology has Property I since the space is separable. However, the associated Sorgenfrey space does not have the property. The double line does not have a G_δ-diagonal. If it had, the space must be metrizable [L]. It is well known that $(R \times \{0, 1\}, \mathcal{T})$ is not metrizable. Hence one of the reasons that $(X, \mathcal{S})$ fails to have Property I is that the ordered space $(X, \mathcal{T})$ does not have a G_δ-diagonal. In the following theorem, metrizability that implies Property I is assumed.

Theorem 1. *If a LOTS $(X, \mathcal{T})$ is metrizable, the associated Sorgenfrey space $(X, \mathcal{S})$ has Property I.*

Proof. There exists a σ-discrete base $\mathcal{B} = \cup\{\mathcal{B}(n) : n \in N\}$ for $(X, \mathcal{T})$ since X is a metrizable LOTS. If $B \in \mathcal{B}(n)$ does not have its maximum, then we choose a point $u(B) \in B$. If $B \in \mathcal{B}(n)$ has its maximum, then let $u(B)$ be the point. Note that $u(B) \in B$. Let $D(n) = \{u(B) : B \in \mathcal{B}(n)\}$ and $D = \cup\{D(n) : n \in N\}$. It is obvious that $D(n)$ is a closed discrete in $(X, \mathcal{S})$. We first show that D is dense in $(X, \mathcal{S})$. Let $[x, y[$ be a non-empty open subset of $(X, \mathcal{S})$. Case (i): If $]x, y[\neq \emptyset$, then there exist a point $z \in]x, y[$ and $B \in \mathcal{B}(n)$ such that $z \in B \subset]x, y[$. Hence $u(B) \in]x, y[\subset [x, y[$. Case (ii): Suppose that $]x, y[= \emptyset$. If x has its predecessor, then $\{x\}$ is open in $(X, \mathcal{T})$. Hence there exists $B \in \mathcal{B}$ such that $x \in B \subset \{x\}$. Hence $B = \{x\}$. Therefore, $u(B) = x$ and $[x, y[\cap D \neq \emptyset$. If x does not have its predecessor, then $]\leftarrow, x]$ is open in $(X, \mathcal{T})$. Hence there exists $B \in \mathcal{B}$ such that $x \in B \subset]\leftarrow, x]$. Hence B has its maximum point. Hence $u(B) = x$. Hence $[x, y[\cap D \neq \emptyset$. This completes the proof.

Corollary 1. *Let* $(X, \mathscr{T})$ *be a LOTS having Property I. If it has a* G_δ*-diagonal, then* $(X, \mathcal{S})$ *has Property I.*

Proof. This follows from Theorem 1 and a fact that a LOTS with a G_δ-diagonal is metrizable [L].

The following theorem states another condition for $(X, \mathscr{T})$ with Property I to assure the same property on $(X, \mathcal{S})$.

Definition 4. Let X be a linearly ordered set and $\{x, y\}$ a two-point subset of X, where $x < y$. If $]x, y[= \emptyset$, then $\{x, y\}$ is said to be a *jump*.

Theorem 2. *Suppose that* $(X, \mathscr{T})$ *is a LOTS with countably many jumps. If* $(X, \mathscr{T})$ *has Property I, then so does* $(X, \mathcal{S})$.

Proof. We first prove that $(X, \mathscr{T})$ has a G_δ-diagonal. Note that a space on which we need to assume is a GO-space and not a LOTS. So we use X instead of $(X, \mathscr{T})$. Let $D = \cup\{D(n): n \in N\}$ be a dense subset of X, where $D(n)$ is closed discrete in X for every $n \in N$. We may assume that $D(n) \subset D(n+1)$. Let $\{\{x_n, y_n\}: n \in N\}$ be countable jumps in X, that means $x_n < y_n$ and $]x_n, y_n[= \emptyset$. Let $D'(n) = D(n) \cup \{x_n, y_n\}$. Then $D'(n)$ is closed discrete in X. Hence we may assume that all jumps are contained in D. Since $X - D(n)$ is open in X, it is expressed as a disjoint union of open convex subsets, that is,

$$X - D(n) = \cup\{U(\alpha) : \alpha \in A(n)\}.$$

Since $D(n)$ is closed discrete and that a GO-space is collectionwise normal, we can find an open set $V(n\ ;\ d)$ for each $d \in D$ such that $V(n\ ;\ d) \cap V(n;\ d') = \emptyset$ for $d \neq d' \in D(n)$. Since X is first-countable, we can find open sets $V(n, m\ ;\ d)\ (m \in N)$ that are contained in $V(n\ ;\ d)$ for $d \in D(n)$ such that $\{V(n, m\ ;\ d): m \in N\}$ is a local base at $d \in D(n)$. Set

$$\mathscr{G}(n, m) = \{U(\alpha): \alpha \in A(n)\} \cup \{V(n, m\ ;\ d): d \in D(n)\},$$

where for $d \neq d'$ in $D(n)$, $V(n, m\ ;\ d) \cap V(n, m\ ;\ d') = \emptyset$ and $V(n, m+1;\ d) \subset V(n, m\ ;\ d)$, $V(n+1, m\ ;\ d) \subset V(n, m\ ;\ d)$ and that $\{V(n, m\ ;\ d) : m \in N\}$ is a countable base at $d \in D(n)$. We show that $\{\mathscr{G}(n, m) : n \in N, m \in N\}$ is a G_δ-diagonal. To show this, x and y are distinct points of X. Case (i): x and y are points of D. There exists $n \in N$ such that $x, y \in D(n)$. There exists $m \in N$ such that $V(n, m\ ;\ x)$ does not contain y. Hence $\mathrm{St}(x, \mathscr{G}(n, m)) = V(n, m)$ does not contain y. Case (ii): Let $x \in D$ and $y \in X - D$. There exists $n \in N$ such that $x \in D(n)$. Suppose that $\{x, y\}$ is a jump, where $x < y$ and $]x, y[= \emptyset$. Hence $x, y \in D$. This is a contradiction. Case (iii): Let $x \in X - D$ and $y \in D$. There exists $n \in N$ such that $y \in D(n)$. Furthermore, there exists $\alpha \in A(n)$ such that $x \in U(\alpha) \subset X - D(n)$. It is easy to see that $U(\alpha)$ does not contain y. If there exists $z \in D(n)$ such that $x \in V(n\ ;\ z)$, then there exists $m \in N$ such that $V(n, m\ ;\ z)$ does not contain y. Hence $\mathrm{St}(x, \mathscr{G}(n, m))$ does not contain y. Case (iv):

Let $x, y \in X - D$. If $]x, y[\cap D \neq \emptyset$, then we can use the argument in Case (iii). Suppose that $]x, y[\cap D = \emptyset$. If $\{x, y\}$ is a jump, then $x, y \in D$. This is also a contradiction. This completes the proof that $\{\mathcal{G}(n, m): n \in N, m \in N\}$ is a G_δ-diagonal. Now let us return to the proof of the theorem. Since $(X, \mathcal{T})$ is a LOTS, it is metrizable by [L]. We then invoke Theorem 1 to get the result. This completes the proof.

The following is worth to note here to give the converse situation to Theorem 2, although it was essentially proved in [BLP].

Proposition 1. *Let X be a GO-space with a G_δ-diagonal. If X has no isolated points, then X has Property I.*

Proof. Let $\{\mathcal{G}(n) : n \in N\}$ be a G_δ-diagonal for X. We assume that each member of $\mathcal{G}(n)$ is an open convex subset of X and that $\mathcal{G}(n+1)$ is a refinement of $\mathcal{G}(n)$. Since X is paracompact by [L, (4.5)], for each $n \in N$, there exists a σ-discrete collection $\{\mathcal{F}(n, m): m \in N\}$ in X, where each member of $\mathcal{F}(n, m)$ is a closed subset of X and $\cup\{\mathcal{F}(n, m): m \in N\}$ is a cover of X for every $n \in N$, and $\mathcal{F}(n, m)$ is a refinement of $\mathcal{G}(n)$. For each $F \in \mathcal{F}(n, m)$, choose a point $p(F) \in F$. Set $D(n, m) = \{p(F): F \in \mathcal{F}(n, m)\}$, then $D(n, m)$ is a closed discrete subset of X. Let $D = \cup\{D(n, m): n \in N, m \in N\}$. Then D is a dense subset of X. To show this, let G be an open set of X and q a point of G. We may assume that G is convex. Since X has no isolated points, there exist at least three points $u < v < w$ in G. There exists $n \in N$ such that $\mathrm{St}(v, \mathcal{G}(n)) \subset]u, w[$. Since $\cup\{\mathcal{F}(n, m): m \in N\}$ is a cover of X, there exist $m \in N$ and $F \in \mathcal{F}(n, m)$ such that $v \in F$. Since there exists $B \in \mathcal{G}(n)$ such that $F \subset B$, v is a point of B. Hence $F \subset B \subset]u, w[\subset G$. Hence $p(F) \in G \cap D$. Therefore, D is a dense subset of X. Hence X has Property I.

4. Dense metrizable subspaces

The following theorem gives an affirmative answer to the question that was posed in [H3].

Theorem 3. *If a LOTS $(X, \mathcal{T})$ has Property II, so does $(X, \mathcal{S})$.*

Proof. Let D be a dense metrizable subspace of $(X, \mathcal{T})$. Then D has a G_δ-diagonal $\{\mathcal{G}(n): n \in N\}$, where every $\mathcal{G}(n)$ is an open cover of D and any two points x and y in D, $x \neq y$, there exists an $n \in N$ such that $\mathrm{St}(x, \mathcal{G}(n))$ does not contain y. Furthermore, we assume that every $G \in \mathcal{G}(n)$ is convex in D. Let $I = \{x : \{x\}$ is open in $(X, \mathcal{S})$ and is not open in $(X, \mathcal{T})\}$. Let

$$D' = D \cap (X - \mathrm{Cl}(I, (X, \mathcal{T}))),$$

where $\mathrm{Cl}(A, X)$ denotes the closure of A in a space X. Let $E = D' \cup I$. Then E is a dense subset of $(X, \mathcal{S})$. To show this, let $x \in X - E$ and $[x, y[$ be a neighborhood of x in $(X, \mathcal{S})$. We first show that $]x, y[\neq \emptyset$. Suppose that $]x, y[= \emptyset$. If x has its predecessor, then $x \in D$ and x does not belong to

Cl(I, $(X, \mathcal{T})$)). Hence $x \in D'$. This is a contradiction. If x does not have its predecessor, then $x \in I$. This is a contradiction. Hence $]x, y[\neq \emptyset$. Since $]x, y[$ is a non-empty open set of $(X, \mathcal{T})$, $]x, y[\cap D \neq \emptyset$. Let $d \in]x, y[\cap D$. If $]x, y[\cap I \neq \emptyset$, then $[x, y[\cap E \neq \emptyset$. Suppose that $]x, y[\cap I = \emptyset$. Then d does not belong to Cl(I, $(X, \mathcal{T})$). Hence $d \in D'$. This is a contradiction. This shows that E is a dense subset of $(X, \mathcal{S})$. Since $X -$ Cl (I, $(X, \mathcal{T})$) is an open subset of $(X, \mathcal{T})$, it is expressed as a union of open convex subsets of $(X, \mathcal{T})$, say, $X -$ Cl (I, $(X, \mathcal{T})) = \cup\{U_\alpha : \alpha \in A\}$, where U_α is a convex component in $(X, \mathcal{T})$. Let $n \in N$, and set

$$\mathcal{H}(n) = \{G \cap U_\alpha : G \in \mathcal{G}(n) \text{ and } \alpha \in A\} \cup \{\{x\} : x \in I\}.$$

We show that $\mathcal{H}(n)$ is an open cover of $(E, \mathcal{S}|_E)$. It is easy to see that $G \cap U_\alpha \subset E$. To show that each member of $\mathcal{H}(n)$ is open in $(E, \mathcal{S}|_E)$, we take $G \cap U_\alpha \in \mathcal{H}(n)$, where $G \in \mathcal{G}(n)$ and $\alpha \in A$. Then $G = V \cap D$, where V is open in $(X, \mathcal{T})$. Since $G \cap U_\alpha \subset D \cap (X - \mathrm{Cl}(I, (X, \mathcal{T}))) = D' \subset E$, we have $G \cap U_\alpha = (V \cap U_\alpha) \cap D = (V \cap U_\alpha) \cap D' = (V \cap U_\alpha) \cap E$, because $(V \cap U_\alpha) \cap I = \emptyset$. Hence $G \cap U_\alpha$ is open in $(E, \mathcal{T}|_E)$. Therefore, $G \cap U_\alpha$ is open in $(E, \mathcal{S}|_E)$. It is obvious that $\{x\} (x \in I)$ is open in $(E, \mathcal{S}|_E)$. To show that $\mathcal{H}(n)$ is a cover of E, let $d \in D'$. Then $d \in D \cap (X - \mathrm{Cl}\ (I, (X, \mathcal{T})))$. Hence there exist $G \in \mathcal{G}(n)$ and $\alpha \in A$ such that $d \in G$ and $d \in U_\alpha$. Hence $d \in G \cap U_\alpha$. Let $d \in I$. Then $d \in \{d\} \in \mathcal{H}(n)$. Finally, we show that $\{\mathcal{H}(n) : n \in N\}$ is a G_δ-diagonal for E. To show that, let x and y are distinct points of E. Let $x \in I$. Since there are no elements $G \cap U_\alpha$ of $\mathcal{H}(n)$ that contain x, it is easy to prove that, for $n \in N$, $\mathrm{St}(x, \mathcal{H}(n)) = \{x\}$ does not contain y. Let $x \in D'$ and $y \in I$. Then there exist $G \in \mathcal{G}(n)$ for (any) $n \in N$ and U_α for some $\alpha \in A$ such that $G \cap U_\alpha$ contains x. Since $G \cap U_\alpha$ has no elements of I, it does not contain y. Note that $\mathrm{St}(x, \mathcal{H}(n)) = \mathrm{St}(x, \mathcal{G}(n)) \cap U_\alpha$. Let x and y be points of D'. Since x and $y \in D$, there exists $n \in N$ such that St $(x, \mathcal{G}(n))$ does not contain y. Since each element of $\mathcal{H}(n)$ is contained in an element of $\mathcal{G}(n)$, $\mathrm{St}(x, \mathcal{H}(n))$ does not contain y. This shows that $\{\mathcal{H}(n) : n \in N\}$ is a G_δ-diagonal for $(E, \mathcal{S}|_E)$. By [BLP, Proposition (3.4)], $(X, \mathcal{S})$ has a dense metrizable subspace. This completes the proof.

5. Property III

We would like to pose the following question that seems to be difficult to answer [H3].

Question. *Let $(X, \mathcal{T})$ be a LOTS having Property III. Does $(X, \mathcal{S})$ have Property III?*

If the assumption on $(X, \mathcal{T})$ is strengthened to Property I, we obtain a result.

Theorem 4. *Suppose that $(X, \mathcal{T})$ has Property I. Then $(X, \mathcal{S})$ has Property III.*

Proof. Let $D = \cup\{D(n) : n \in N\}$ be a dense subset of $(X, \mathcal{T})$ such that $D(n)$ is a closed discrete subset of $(X, \mathcal{T})$ for every $n \in N$. Let $U(0) = D(0)$ be the set of isolated points of $(X, \mathcal{S})$. For every $n > 0$, set $U(n) = X$. It is clear that, for every $n \geq 0$, $U(n)$ is open in $(X, \mathcal{S})$ and $D(n)$ is relatively closed discrete in $U(n) \subset (X, \mathcal{S})$. Then the collection $\{U(n), D(n) : n \geq 0\}$ builds what is necessary to assure Property III. To see this, let p be a point of X and G an open subset of $(X, \mathcal{S})$ containing p. First let p be an isolated point of $(X, \mathcal{S})$. Then $p \in U(0)$ and $G \cap D(0) \neq \emptyset$. Suppose that p is not an isolated point of $(X, \mathcal{S})$. Then we may assume that $G = [p, q[$ and $]p, q[\neq \emptyset$, where $p < q$. Since $]p, q[$ is open, and D is dense, in $(X, \mathcal{T})$, it follows that $]p, q[\cap D \neq \emptyset$. Hence there exists an $n \geq 1$ such that $]p, q[\cap D(n) \neq \emptyset$ and $p \in U(n) = X$. Hence $G \cap D(n) \neq \emptyset$. This completes the proof of Theorem 4.

References

[BL] H. R. Bennett and D. J. Lutzer, *Point countability in generalized ordered spaces*, Top. and its Appl., **71** (1996), 149-165.

[BLP] H. R. Bennett, D. J. Lutzer and S. D. Purisch, *On dense subspaces of generalized ordered spaces*, Top. and its Appl., **93** (1999), 191-205.

[H1] M. Hosobuchi, *Property III and metrizability in linearly ordered spaces*, J. of Tokyo Kasei Gakuin Univ., **37** (1997), 217-223.

[H2] M. Hosobuchi, *Relationships between four properties of topological spaces and perfectness of generalized ordered spaces*, Bull. of Tokyo Kasei Gakuin Tsukuba Women's Univ., **2** (1998), 67-76.

[H3] M. Hosobuchi, *Properties concerning dense subsets of Sorgenfrey spaces and Michael spaces*, Bull. of Tokyo Kasei Gakuin Tsukuba Women's Univ., **3** (1999), 233-241.

[L] D. J. Lutzer, *A metrization theorem for linearly orderable spaces*, Proc. Amer. Math. Soc., **22** (1969), 557-558.

Tokyo Kasei Gakuin University
Department of Housing and Planning
2600 Aihara, Machida, Tokyo 194-0292
JAPAN
mhsbc@kasei-gakuin.ac.jp