A symmetry-based mass formula for non-strange nuclei and Λ hypernuclei

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We describe a symmetry-based extension of the Weizsäcker mass formula, which is inspired by the spin–flavour SU(6) symmetry. We apply this formula for the simultaneous description of normal (i.e. non-strange) nuclei and Λ hypernuclei and calculate binding and separation energies. We also compare our formula with another similar extension of the Weizsäcker mass formula to Λ hypernuclei.

1. Introduction

The available experimental information on nuclear masses has recently reached a new dimension, namely, that of the strangeness. The number of known hypernuclei is not negligable anymore, and their masses have also been determined.[1] Therefore we are faced with the question of describing them together with the masses of the normal isotopes.

In addition to this very natural requirement, there is another, much more exotic aspect of the binding energy problem which is the question of the stability of the strange matter. The efforts have been concentrated so far mainly on this aspect. One can inquire about the stability with respect to the strong interaction, as well as with respect to the weak decay; the latter one would result in an absolutely stable strange matter. The question can be addressed on the level of quark matter, or on the level of hadronic matter. Concerning the strange quark matter, there are predictions even for absolute stability, i.e. with respect both to the strong and to the weak interactions.[2] As for the strange hadronic matter, it was predicted only recently[3] that with respect to strong decay it can also be stable.

These considerations on stability are, of course, directly related to the masses, or binding energies, therefore in these investigations the calculation of masses plays a crucial role. Such studies have been carried out within the framework of the relativistic mean field approach[4] and of the Fermi gas model combined with phenomenologically determined one-boson-exchange model interactions between the barions.[3] An important characteristic feature of the partially or absolutely stable strange matter is the large strangeness ($|\mathcal{S}|/A \approx 0.5$–$1.0$) and small electric charge ($Q/A \approx 0.1$).

The experimentally available information on strange matter, i.e. the known hypernuclei represent a considerably different situation: the strangeness is small, and the electric charge is large. In fact, most of the known hypernuclei contain a single Λ hyperon. Therefore other approaches to the hypernuclear masses seem to be needed as well, in which the ground-state-like nuclear structure effects are also taken into account. Recently we have proposed such an approach to Λ hypernuclei, based on a symmetry-based extension of the Weizsäcker mass formula.[5,6] The symmetry is that of the flavour-spin SU(6) obtained as a combination of the isospin-strangeness SU(3) and spin SU(2) symmetries. It is a natural enlargement of Wigner's spin-isospin SU(4)[7] symmetry, first proposed by Gürsey and Radicati[8] in hadron spectroscopy.

In this contribution we present the SU(6) symmetry-based mass formula and its application for the description of the binding energies of normal isotopes and Λ hypernuclei.

The structure of the paper is as follows. In Section 2 we discuss how the Majorana operator (or the invariant operator of the spin-isospin symmetry group SU(4)) can substitute the usual pairing term in the mass formula of normal nuclei. In Section 3 the scenario is generalised by incorporating the strangeness degree of freedom. In Section 4 the mass formula is applied simultaneously to normal isotopes and Λ hypernuclei, by calculating binding energies and nucleon, as well as Λ separation energies. In Section 5 a comparison is made with the mass formula of Dover and Gal,[3] which also treats the binding energy of the Λ-hypernuclei in an extended Weizsäcker-type formula. Finally, Section 6 contains the conclusions.

2. Non-strange nuclei and the spin–isospin symmetry

It was shown in Ref. 9 that an adequate nuclear mass formula is obtained by adopting the usual volume, surface, Coulomb and asymmetry terms but replacing the pairing term with the expectation value of the space-exchange or Majorana operator. This leads to the following expression for the binding energy $B(N,Z)$ of a nucleus with N neutrons, Z protons and $A = N + Z$ nucleons:

$$\begin{aligned} B(N,Z) &= a_{\rm v}A - a_{\rm s}A^{2/3} - a_{\rm c}\frac{Z^2}{A^{1/3}} \\ &\quad -a_{\rm a}\frac{(N-Z)^2}{A} + a_{\rm m}\frac{\langle\hat{M}\rangle}{A^{\gamma_{\rm m}}}. \end{aligned} \quad (1)$$

The Majorana operator is defined as $\hat{M} = \sum_{i<j=1}^{A} \hat{M}_{ij}$ where $\hat{M}_{ij}$ interchanges the spatial coordinates of particles i and j,

$$\begin{aligned} &\hat{M}_{ij}\phi(\vec{r}_1,\ldots,\vec{r}_i,\ldots,\vec{r}_j,\ldots,\vec{r}_A) = \\ &\qquad \phi(\vec{r}_1,\ldots,\vec{r}_j,\ldots,\vec{r}_i,\ldots,\vec{r}_A), \end{aligned} \quad (2)$$

and thus has the eigenvalue $+1$ in a space-symmetric state of two nucleons and -1 in an antisymmetric state. Due to the antisymmetry of the total wavefunction, the spatial symmetry of a state ('measured' by $\hat{M}$) determines its spin–isospin or U(4) symmetry[7]: the spatial and spin–isospin parts of the wavefunction are characterised by Young diagrams that are adjoint to each other. The U(4) Young diagram $[f_1 f_2 f_3 f_4]$ characterising a nucleus can readily be determined remembering that the f_i correspond to the four single-particle states (proton and neutron, spin-up and spin-down) the nucleons can occupy. These f_i also determine the SU(4) labels $(\lambda\mu\nu)$ used in Ref. 9.

The rationale behind the use of the expectation value $\langle\hat{M}\rangle$ in Eq. (1) is the following. Due to the short-range attractive nature of the residual nuclear interaction, nucleons in the nucleus attempt to maximise their spatial symmetry and choose a specific, favoured U(4) (and therefore SU(4)) representation. For a given nucleus the expectation value of $\langle\hat{M}\rangle$ in Eq. (1) is then

$$\langle\hat{M}\rangle = -\frac{1}{2}\sum_{i=1}^{4} f_i(f_i + 1 - 2i). \tag{3}$$

The SU(4) symmetry, induced by the short-range character of the residual nuclear interaction, is broken by the spin–orbit term in the nuclear mean field and increasingly so in heavier nuclei. Therefore a mass dependence $A^{-\gamma_{\rm m}}$ is included in Eq. (1) and it is expected that $\gamma_{\rm m} > 0$.

From a pragmatic point of view the mass formula in Eq. (1) is preferable to the usual one since for all regions of nuclei it gives a comparable or lower root-mean-square (rms) deviation. For example, a fit with Eq. (1) to the 1909 known nuclear masses[10] yields an rms deviation of 2.68 MeV while the usual Weizsäcker formula gives 3.46 MeV. The role of the Majorana term in Eq. (1) is that it provides an approximate but adequate description of two nuclear correlation effects: pairing (it distinguishes even–even from odd-mass and odd–odd nuclei) and α-particle correlations which are of particular importance in $N \sim Z$ nuclei,[11] but also seem to persist in heavy nuclei.

Finally, we note that in Ref. 9 the Casimir operator of SU(4) was used instead of the Majorana operator $\hat{M}$ which is used here. The latter term allows an easier extension to hypernuclei and generally yields also somewhat lower rms deviation. Also, the terminology there was based on the use of the SU(4), rather than the U(4) labels. See also Ref. 12.

3. Λ hypernuclei and the SU(6) symmetry

Guided by the attempts of the previous section a mass formula for the simultaneous description of normal nuclei and Λ hypernuclei can now be obtained as follows.[5] Wigner's SU(4) symmetry arises as a result of the combined invariance in spin and isospin. If the strangeness degree of freedom is included, isospin $\mathrm{SU}_T(2)$ is replaced by flavour $\mathrm{SU}_F(3)$ and the combined spin–flavour invariance gives rise to the SU(6) classification of Gürsey and Radicati[8] which can be summarised with the following scheme:

$$\mathrm{U}(6) \supset (\mathrm{SU}_F(3) \supset \mathrm{U}_Y(1) \otimes \mathrm{SU}_T(2)) \otimes \mathrm{SU}_S(2). \tag{4}$$

The results described here are obtained by taking $(\lambda\mu) = (10)$ as the fundamental representation of $\mathrm{SU}_F(3)$ which includes the neutron, proton and Λ hyperon. This corresponds to the SU(3) model of Sakata.[13]

Originally, the Sakata model was proposed for the explanation of the hadron spectrum. It turned out to be incorrect; the building blocks of the hadrons are quarks, rather than neutrons, protons and Λ hyperons. For Λ hypernuclei, however, these particles are the natural building blocks and therefore the application of a 'nuclear Sakata model' for their description seems a justified choice.

The generalisation of (1) to the binding energy $B(N,Z,\Lambda)$ (where Λ denotes the number of Λ hyperons) is

$$\begin{aligned} B(N,Z,\Lambda) &= a_{\rm v}A - a_{\rm s}A^{2/3} - a_{\rm c}\frac{Z^2}{A^{1/3}} - a_{\rm a}\frac{(N-Z)^2}{A} \\ &\quad + a_{\rm y}\frac{\mathcal{S}}{A^{\gamma_{\rm y}}} + a_{\rm m}\frac{\langle\hat{M}\rangle}{A^{\gamma_{\rm m}}}, \end{aligned} \tag{5}$$

where $A = N + Z + \Lambda$ and the strangeness is $\mathcal{S} = Y - B$ with B the baryon number and Y the hypercharge. The effect of the strangeness is represented by a simple linear term, but it is also implicitly contained in the Majorana term. To reduce the number of parameters an equal mass dependence $\gamma_{\rm y} = \gamma_{\rm m}$ is assumed. The Majorana operator is related to the the Casimir invariants of SU(6) and SU(4) through the general formula

$$\hat{M} = -\frac{1}{2N}\left(A(A - N^2) + \hat{C}_2[\mathrm{SU}(N)]\right). \tag{6}$$

This formula recovers the known result for SU(4) with the $N = 4$ choice[12] and direct calculation proves its validity for $N = 6$ too. The eigenvalues of $\langle\hat{M}\rangle$ have a form similar to (3) (but with $i = 1, \ldots, 6$), which secures the equivalence of Eqs. (1) and (5) for normal nuclei.

Application of Eq. (5) requires in addition the knowledge of the favoured SU(6) (or U(6)) representation. This can be found again by the hypothesis of maximum spatial symmetry of the nuclear wavefunction. This yields, for normal nuclei, the favoured U(6) representation $[f_1 f_2 f_3 f_4 00]$ with f_i determined in the U(4) scheme, as in the previous Section. For nuclei with one Λ hyperon in the s shell, a one-boxed row is added to the U(4) Young diagram while for nuclei with two Λs in the s shell two one-boxed rows are added. This results in the U(6) representations $[f_1 f_2 f_3 f_4 10]$ and $[f_1 f_2 f_3 f_4 11]$, respectively, unless f_4 (or f_3, etc.) is zero, which happens only for the lightest nuclei.

The Weizsäcker mass formula and its extensions contain contributions originating from different nuclear models. The novel feature of the symmetry-based formulae in Eqs. (1) and (5) is that many-particle correlations are included on the basis of the SU(4) and SU(6) models, instead of considering purely empirical terms. This, however, does not mean that these symmetries are conserved by all interactions: the Coulomb and the asymmetry terms, for example, break them. In the SU(6) case a further symmetry breaking is added, which is represented by $\mathcal{S}$, the Casimir invariant of $\mathrm{U}_Y(1)$. This SU(6)-symmetry breaking is related to the different nature of the nucleon–nucleus and Λ–nucleus interaction and is of the same form as the one appearing in the Gell-Mann–Okubo mass formula.[14)]

4. Applications

4.1 Binding energies

To illustrate the performance of the mass formula in Eq. (5), available data have been analysed for the binding energy of 1909 normal nuclei, 35 single-Λ hypernuclei and three double-Λ hypernuclei.[5)] The results are given in Table 1. Four types of fits were carried out which differed in the nuclei included in the fit. In each case fits were performed to normal nuclei (N), to hypernuclei (H) and to normal and hypernuclei (N + H).

Table 1. Coefficients in the mass formula (5) and associated rms deviations.*

Fit	a_v	a_s	a_c	a_a	a_y	a_m	$\gamma_y = \gamma_m$	rms		
								N	H	N+H
All nuclei (1947 of which 38 hypernuclei)										
N	22.88	35.30	0.60	13.09	—	14.07	0.79	2.68	—	—
H	19.33	27.37	0.45	14.65	21.11	6.11	0.70	—	2.44	—
N+H	22.63	34.68	0.60	13.44	60.19	13.58	0.79	2.68	4.41	2.72
$N, Z \le 20$ (228 nuclei of which 33 hypernuclei)										
N	23.22	36.90	0.49	9.20	—	16.24	0.81	2.49	—	—
H	21.55	32.38	0.67	13.01	6.74	4.60	0.59	—	2.19	—
N+H	20.92	31.30	0.48	10.91	47.99	11.70	0.80	2.56	3.41	2.70
$2 \le N, Z$ (1937 nuclei of which 35 hypernuclei)										
N	22.73	34.93	0.60	13.36	—	13.74	0.79	2.62	—	—
H	24.93	41.42	0.41	6.97	73.82	16.29	0.75	—	2.10	—
N+H	23.62	36.95	0.61	13.03	84.52	15.96	0.80	2.62	3.19	2.63
$2 \le N, Z \le 20$ (218 nuclei of which 30 hypernuclei)										
N	27.58	47.54	0.51	6.67	—	20.68	0.77	2.14	—	—
H	26.44	44.53	0.64	6.13	60.30	14.99	0.73	—	1.88	—
N+H	26.52	44.95	0.51	7.37	88.35	19.06	0.77	2.14	2.10	2.14

*All quantities are in MeV, except $\gamma_y = \gamma_m$ which is dimensionless.

It is seen that the separate fits N and H generally yield close coefficients which in consequence are close to those obtained in the combined fit N + H. The lowest rms deviations occur for the H fits. However, low rms deviations in the H fits are not necessarily the best indications for the success of Eq. (5). A more telling sign is having comparable rms deviations for normal and hypernuclei in the N + H fits, because this signifies that the two types of nuclei can be treated on an equal footing. In this respect the fit in the $2 \le N,\ Z \le 20$ domain (containing the majority of known hypernuclei) is the most successful and the N and N + H parameter sets are also closest to each other. The $2 \le N, Z$ fit is worse, and the explanation for this is the following (see Ref. 5). The increased rms deviation is due to hypernuclei close to shell closures, like ${}^{56}_{\Lambda}$Fe and ${}^{208}_{\Lambda}$Pb. However, the binding energies of the neighbouring normal nuclei have comparable deviations, which shows that the shell structure has similar effect on normal and Λ hypernuclei, justifying again their unified treatment in terms of Eq. (5). (See Fig. 1 in Ref. 5.)

The difference between certain coefficients in the N, H and N + H fits might be the consequence of the relative importance of the corresponding terms for the different types of nuclei in the given mass regions. Leaving out medium and heavy nuclei, for example, results in comparable values of a_a for the three types of fits. This is because the hypernuclear data set is dominated by light nuclei, where $(N-Z)^2/A$ is small, and therefore the asymmetry term does not have too large a contribution to the binding energy, while normally the fits include mainly medium and heavy nuclei, with large values of $(N-Z)^2/A$. Leaving out nuclei with small A yields more uniform values for a_m. Here the explanation is that the Weizsäcker mass formula is not designed for the lightest nuclei, so it is reasonable to exclude them from the fits. We also note that the linear dependence of Eq. (5) on $\mathcal{S}$ seems justified, since the binding energies of the three double-Λ hypernuclei (${}^{6}_{\Lambda\Lambda}$He, ${}^{10}_{\Lambda\Lambda}$Be and ${}^{13}_{\Lambda\Lambda}$B) are reasonably reproduced, even when they are not included in the fit.

A remarkable result is that the A dependence of the Majorana term is close to that of the pairing term ($A^{-3/4}$) of the usual Weizsäcker mass formula: γ_m is between 0.7 and 0.8 in most fits. In fact, closer inspection reveals that the Majorana term introduces a splitting between even–even, odd–even and odd–odd (normal) nuclei similar to the pairing term. This confirms the finding that the Majorana term can be considered as a sophisticated replacement of the pairing term, the generalisation of which to hypernuclei would otherwise be somewhat problematic, as one has to deal with three types of constituents (neutron, proton and Λ) instead of just two. In fact, rewriting the expectation value of the Majorana operator one finds that the resulting terms are similar to some expressions (e.g. pairing and Wigner terms) used frequently in Weizsäcker-type mass formulae. (See Eq. (8) in Ref. 5.)

In Ref. 5 we showed that the shell effects clearly show up in our results as bumps in the $\Delta B \equiv B(N,Z,\Lambda)_{\rm expt} - B(N,Z,\Lambda)_{\rm th}$ curve at the standard magic numbers. Therefore here we focus on quantities, which are less sensitive to these effects.

Inspecting the $B(N,Z,0)_{\rm th}/A$ function along the valley of stability we find that apart from some minor deviations for the lightest nuclei, the curve follows the experimental trends. The peak of this curve is found to be at the same mass number ($A = 62$) as that known from the experimental data, furthermore, it coincides with the most strongly bound nuclid, ^{62}Ni. Comparing the prediction of our formula for each A-chain, we find that for the majority of A values the nuclids predicted to be most strongly bound coincide with the nuclids found at the bottom of the valley of stability. There are altogether 97 exceptions, with $|\Delta Z| = 1$ and 2 in 72 and 25 instances respectively, mainly for heavy nuclei.

4.2 Separation energies

We have calculated the proton, neutron and Λ separation energies according to the formulae $S_p = B(N,Z,0) - B(N,Z-1,0)$, $S_n = B(N,Z,0) - B(N-1,Z,0)$, $S_\Lambda = B(N,Z,1) - B(N,Z,0)$. (We note that we used these formulae for the experimental proton and neutron separation energies too, although they are also available directly in the literature.[15)]) The results are presented in Figs. 1, 2 and 3. In the model calculations we used the parameter set obtained by fitting our mass formula to normal and hypernuclei with $N,\ Z \ge 2$, denoted with N + H in Table 1. This parameter set seems to be most appropriate for predictions throughout the whole range of the mass number A: it includes all the nuclei, except for the lightest ones, which are not expected to be describable

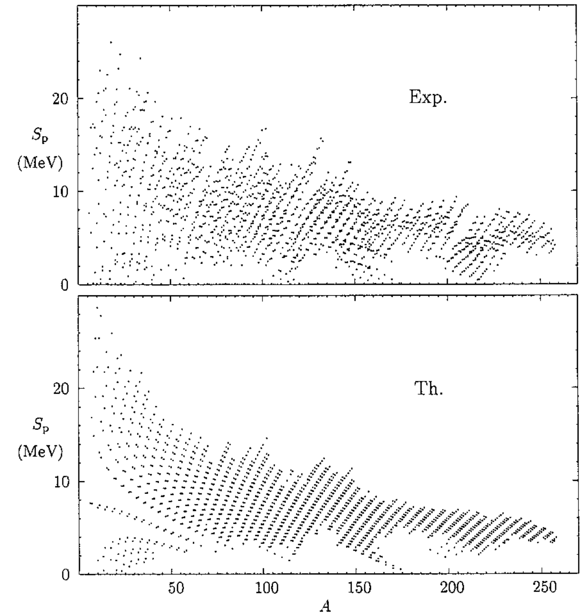

Fig. 1. Proton separation energies $S_p = B(N, Z) - B(N, Z-1)$ taken from experiment (upper panel) and from calculations (lower panel).

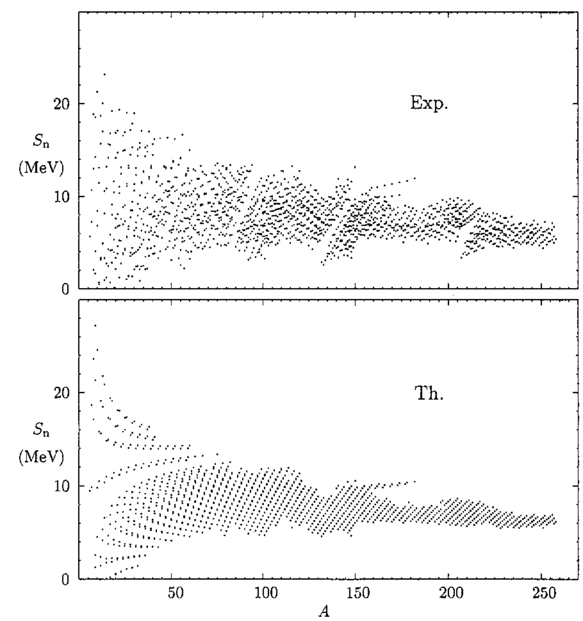

Fig. 2. Neutron separation energies $S_n = B(N, Z) - B(N-1, Z)$ taken from experiment (upper panel) and from calculations (lower panel).

in terms of the Weizsäcker mass formula and its generalisations.

As it can be seen from Figs. 1 and 2, our mass formula reproduces the gross features of the nucleon separation energies for normal nuclei. The "striped" structures tilted to the right from the horizontal direction correspond to N-chains and Z-chains (with Z and N fixed), respectively. This feature characterises both the theoretical and the experimental plots, but is somewhat more pronounced for the proton separation energies in the latter case. The gaps between these stripes in the experimental plots near the mass region of

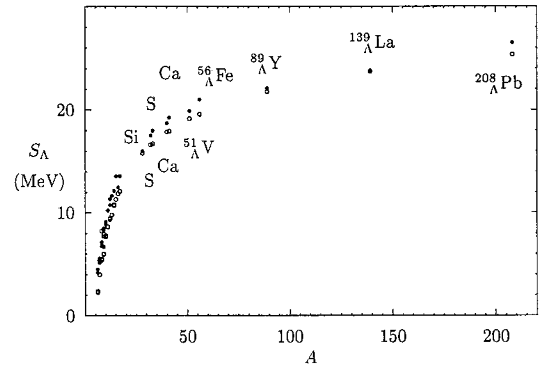

Fig. 3. Λ separation energies $S_\Lambda = B(N, Z, 1) - B(N, Z, 0)$ taken from experiment (full circles) and from calculations (open circles). The known hypernuclei are also indicated for $20 < A$: up to $A = 50$ only the name of the given element is displayed, while for the heavier nuclei the whole name is given.

magic nuclei are clearly shell effects. These gaps are more pronounced for the neutron case (Fig. 2) and occur near $A = 70$, 130 and 210, but are also visible to some extent in the proton case (Fig. 1) near $A = 210$, for example.

The theoretical plots also exhibit further systematic structures, which are less obvious in the experimental plots. Neighbouring isotope and isotone chains tend to occur in parallel pairs, for example, which is the manifestation of a pairing effect predicted by the Majorana term. Also, there are stripe-like structures roughly perpendicular to the N and Z-chains: these often represent chains of the type (N, Z), $N-2$, $Z-2, \ldots$. Similar structures can also be identified in the experimental plots. (See, e.g. the "peninsula" at $A \simeq 160$ and $S_p \simeq 0.5$ MeV in Fig. 1, and at $A \simeq 170$ and $S_n \simeq 10$ MeV in Fig. 2.

As expected, the highest separation energies are found at the closure of the respective nucleon shell, while the lowest ones typically occur just off this nucleon (proton or neutron) number. Although shell effects are not contained in our mass formula, the calculated nucleon separation energies also follow this pattern. (We omitted some data points from the theoretical plots in cases when negative separation energies were obtained. This happened in 5 and 7 cases for S_p and S_n, respectively.)

One important feature of our symmetry-based mass formula is that one can describe normal and hypernuclei on an equal footing in terms of it. According to this, Λ separation energies can also be obtained in a similar straightforward way. Figure 3 displays the experimental and the calculated S_Λ values for the known single-Λ hypernuclei. (Due to the low number of data points, we display these quantities on a single plot. We also note that we omitted the $^{5}_{\Lambda}$He nucleus, which was predicted to be unbound.)

The trend of the calculated points follows that of the experimental ones. We note that the error of the experimental data points ranges between 0.5 and 1.5 MeV for $A > 30$. There is, however, a trend which indicates that the pairing effect coming from the Majorana operator might not be completely correct for all mass regions: for $A > 50$ the experimental and the theoretical dots are closer to each other for odd values of

A ($A = 51,\ 89,\ 139$), while for $A < 50$ the situation seems to be the opposite ($A = 28,\ 32,\ 40$).

5. Comparison with another mass formula

Our mass formula in Eq. (5) was designed to describe the available experimental data on normal and hypernuclei. We considered the coefficients appearing in it as parameters to fit and did not relate them to physical quantities derived from some models. A different approach is used by Dover and Gal,[3] who constructed another generalisation of the Weizsäcker mass formula to Λ (and other) hypernuclei. In their equation the binding energy is

$$B^{(\mathrm{DG})}(N, Z, \Lambda) = a_{\mathrm{v}}A + b_{\mathrm{v}}yA - a_{\mathrm{s}}A^{2/3} -a_{\mathrm{c}}\frac{Z^2}{A^{1/3}} - a_{\mathrm{x}}x^2A - b_{\mathrm{y}}y^2A\ , \qquad (7)$$

where $x = (N - Z)/A$ and $y = [(N + Z)/2 - \Lambda]/A$ are the neutron and nucleon excess ratios. It is clear that the conventional volume term, the surface term, the Coulomb term and the asymmetry term are common for Eq. (7), Eq. (5) and the usual Weizsäcker formula. Eq. (7) does not contain the pairing term of the Weizsäcker formula, and the generalisation to Λ hypernuclei is done by introducing an additional volume ($b_{\mathrm{v}}yA$) and asymmetry ($b_{\mathrm{y}}y^2A$) term. In Eq. (5) this is taken care of by the Majorana and the strangeness terms. As opposed to our case, the authors of Ref. 3 derived the coefficients of this formula from arguments based on the Fermi gas model of the nucleus combined with phenomenologically determined interactions.

In constructing this formula, Dover and Gal aimed at describing strange hadronic matter and wanted to test its stability with respect to strong decay. Therefore they supposed that the strangeness ratio (therefore the relative number of Λ hyperons) is large. The main merit of this approach is its predictive power concerning the strange hadronic matter, nevertheless, at the same time Eq. (7) is also a mass formula for Λ hypernuclei, and as such, it can be used for the phenomenological description of the experimentally known binding energies. Thus besides the qualitative comparison of Eqs. (7) and (5), a quantitative analysis can also be useful to establish a relation between the two models. This is especially so, if we consider that to our best knowledge, no other similar extensions of the Weizsäcker mass formula are available besides these two approaches. Therefore we performed fit of the parameters of Eq. (7) in the same spirit as we did for Eq. (5) in Section 4.

In particular, we considered the cases denoted with $\mathrm{N} + \mathrm{H}$ in Table 1, i.e. we performed simultaneous fits for the known normal and hypernuclei in various mass domains. The resulting parameter sets are displayed in Table 2. There are several comments in order. One is that the rms values of the deviation from the experimental data are usually higher (typically with 1 MeV) than the corresponding $\mathrm{N} + \mathrm{H}$ data in Table 1. This is not surprising, because Eq. (5) has one more parameter to fit ($\gamma_{\mathrm{y}} = \gamma_{\mathrm{m}}$), and also there is nothing to account for pairing in Eq. (7). Another interesting feature is that excluding the lightest nuclei, the rms deviation for the hypernuclei does not change much for Eq. (7), as opposed to the case of Eq. (5). Therefore the rms deviation for the normal and the hypernuclei is less uniform for Eq. (7) than for Eq. (5). (See e.g. the case of $2 \le N,\ Z \le 20$, $\mathrm{N} + \mathrm{H}$ in Tables 1 and 2, for example.)

Table 2. Coefficients obtained from fitting the mass formula in Eq. (7) to the available normal and **Λ** hypernuclear masses. The notations are the same as in Table 1. All quantities are in MeV.

Fit	a_{v}	a_{s}	a_{c}	a_{x}	b_{v}	b_{y}	rms N	rms H	rms N+H
All nuclei (1947 of which 38 hypernuclei)									
N+H	14.51	16.75	0.70	22.69	7.19	10.54	3.55	4.68	3.57
$N, Z \le 20$ (228 nuclei of which 33 hypernuclei)									
N+H	13.27	14.59	0.46	16.01	6.47	9.61	3.11	4.59	3.37
$2 \le N, Z$ (1937 nuclei of which 35 hypernuclei)									
N+H	15.00	16.86	0.70	22.78	5.21	8.40	3.45	4.68	3.48
$2 \le N, Z \le 20$ (218 nuclei of which 30 hypernuclei)									
N+H	14.06	14.78	0.48	16.46	3.20	5.79	3.01	4.37	3.23

It is also remarkable that the parameters obtained for the fits which included also the medium and heavy nuclei (see the first and the third lines of Table 2) are rather close to one of the two standard parameter sets used in Ref. 3. This is the parameter set denoted by I in Ref. 3 (in MeV): $a_{\mathrm{v}} = 18.4$, $a_{\mathrm{s}} = 16.9$, $a_{\mathrm{c}} = 0.72$, $a_{\mathrm{x}} = 28.5$, $b_{\mathrm{v}} = 7.6$ and $b_{\mathrm{y}} = 25.1$. The largest differences with respect to the parameters in Table 2 occur for a_{x} and b_{y}, the parameters related to the neutron and nucleon excess. We note that the corresponding terms, especially the second one are most sensitive to the strangeness ratio of the nuclei. The alternative parameter set, denoted with II differs from I only in the volume terms: $a_{\mathrm{v}} = 13.4$, $b_{\mathrm{v}} = 17.6$. Although the a_{v} parameter of this set is rather close to that found in the fitting procedure, the corresponding b_{v} gets too large at the same time. We note that parameter sets I and II in Ref. 3 differ in the Λ-Λ interaction used; in particular it is zero for set II.

Calculating the rms values calculated with the parameter sets I and II directly we get rather high values, typically around 20 MeV. The situation is more favourable though, for set II, where at least the rms value for the hypernuclei (denoted with H) is relatively small, 12.26 MeV, compared to the other rms values (N and $\mathrm{N} + \mathrm{H}$).

There is another mass formula presented in Ref. 3 especially for single-Λ hypernuclei. There the two strangeness terms are replaced with a single one of the form $A^{-2/3}\delta_{\Lambda,1}$. (See Eq. (A.5) in Ref. 3.) The parameter of this term is kept at the constant value of 120 MeV. Fitting directly the remaining parameters of this formula leads to qualitatively similar results as that of the fit of Eq. (7).

6. Conclusion

In the present contribution we have described an extension of the Weizsäcker mass formula for the simultaneous description of normal nuclei and Λ hypernuclei. This new mass formula, inspired by the SU(6) spin–flavour symmetry, reduces for non-strange nuclei to one based on Wigner's SU(4) spin–isospin symmetry. The conventional pairing term in the Weizsäcker formula can be successfully replaced by a Majorana term which chooses maximal spatial symmetry for the nucleons and Λ hyperons as a consequence of the short-range attractive nature of the residual nuclear interaction. Fits to normal nuclei and Λ hypernuclei, separately as well as simultaneously, yield rms deviations that are comparable in each case.

Calculations were presented for the standard properties related to nuclear masses, such as binding energies, nucleon and Λ separation energies and the location of the valley of stability. Also, comparison was made with the only known similar extension of the Weizsäcker mass formula, in which hypernuclear masses are discussed on the hadronic level together with the masses of normal nuclei.

Given the flexible nature of algebraic models, generalisations of the proposed mass formula can be envisaged. For example, quark degrees of freedom can be included by considering colour singlets in a $\mathrm{U}(18) \supset \mathrm{U}_C(3) \otimes \mathrm{U}_{FS}(6)$ colour+spin–flavour classification. This scheme leads, for three quarks, to the octet and decuplet of $\mathrm{SU}_F(3)$ and conceivably can be used to construct a mass formula for a wider class of hypernuclei which involve, besides Λ, also Σ, Ξ, Δ, Σ^*, Ξ^* and Ω particles. A further possibility consists of incorporating shell effects in the mass formula, which can also be aided by algebraic techniques. With its more accurate treatment of the spin–isospin degrees of freedom the present mass formula offers an appropriate starting point for such studies.

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