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SHELF SPACE ALLOCATION WITH LOCATION EFFECTS ON DEMAND

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ABSTRACT

This paper addresses a problem of retailer who sells various brands of items through displaying on multi-level shelves. We formulate the shelf space allocation problem in a mathematical model with the objective of maximizing the retailer's profit. Then a solution procedure is developed based on the gradient search. The validity of the model is illustrated with an example problem.

INTRODUCTION

Shelf on which products are being displayed is one of the most important resources in retail environment. Accordingly, shelf management has been considered as an important decision to retailers. Retailers can not only increase their profit but also decrease cost by managing shelf well. From Dreze et al. [6], shelf management to increase sales by attracting the consumer's attention and encouraging consumers to have additional purchase opportunities can be implemented by manipulating following; 1) shelf space; 2) products display (location of the product within a display, product adjacencies, aesthetic elements etc.).

Many research articles [1, 2, 3, 4, 5, 7] appeared on the shelf space allocation problem to deal with how to optimally allocate shelf space to each brand of items so as to maximize the total sales volume. These models formulated the demand rate as a function of the shelf space allocated to products. However, they did not consider the location effects that Dreze et al. [6] emphasized. Yang [9] proposed a space allocation model incorporating the main effects and the cross effects of demand as well as the location effects similar to multi-constraint knapsack problem.

In this paper, we develop the shelf space allocation model integrated with inventory-control models considering location effects. The reminder of this paper is organized as follows. First, we develop a mathematical model for shelf space allocation problem based on 'full shelf merchandising policy'. And then a solution procedure is developed based on the gradient search. Finally, validity of the model is illustrated with example problems and solutions are compared with those obtained from a total enumeration.

NOMENCLATURE

L_j: the space of shelf X_{ij}
L_B: backroom space capacity
l_i: the amount of shelf space required for a unit of item i
P_i: unit selling price of item i
Cp_i: unit purchasing cost of item i
Ch_i: inventory holding cost of item i per unit period
Cs_i: shelf space cost per unit period based on the space allocated to item i
Co_i: ordering cost of item i per order
$X_i^{\min}$: minimum number of item i that can be displayed on shelves, $X_i^{\min} > 0$
$X_i^{\max}$: maximum number of item i, which can be displayed on shelves
$Q_i^{\min}$: minimum ordering quantity of item i required from supplier, $Q_i^{\min} > 0$
$Q_i^{\max}$: maximum ordering quantity of item i required from supplier
T_i: replenishment cycle time of item i
X_{ij}: the number of item i displayed on shelf j
X_i : total number of item i displayed on shelves = $\sum_{j=1}^{M} X_{ij}$
Q_i: order quantity of item i

d_i : the scale parameter of demand function for item i, $d_i>0$
β_i: the space elasticity for item i with respect to a unit of displayed inventory, $0 < \beta_i < 1$
β_{ik}: the cross space elasticity between item i and k
α_j: the scale parameter that reflects the increase of demand rate with respect to the level of shelf when items are displayed on shelf j
a_i: weighted average value of α_j when item i is displayed split among more than one shelve

MODEL DEVELOPMENT

We deal with the problem of a retailer who displays various brands of items within a category to multi-level shelves that have limited space. Orders are initially received into backroom that is placed in the back of the multi-level shelves, and the items are restocked from the backroom into the shelves as the displayed items on shelves are depleted by consumer demand. The retailer's total profit can be expressed as the gross margin subtracted by the holding cost of the items in the backroom and those on display, display expense, and ordering cost. We want to determine how many items to order for each brand, how much space to allocate to each brand and where each brand to be placed on shelves in order to maximize the retailer's total profit.

[Assumptions]

[1] The system involves N brands of items within a category and M shelves with limited space and a limited backroom capacity.
[2] Demand rate of item is a function of the quantity displayed and the location displayed within the shelves.
[3] The time horizon of the inventory model is infinite.
[4] Replenishments to the system are independent for each item (no joint replenishments), sent directly to the backroom inventory, and instantaneous with a known and constant lead-time.
[5] The retailer order items with (r, Q) policy.
[6] "Full-shelf merchandising" policy is adopted suggested by Larson and DeMarais [8]; that is, the display area is always kept fully stocked.
[7] All N brands of items have to be displayed on shelves, i.e., N brands of items are included in the assortment.
[8] The items are restocked from the backroom into the shelves continuously and the restocking cost is negligible.
[9] The selling price and the unit cost of each item are known and constant.

We focus on the experimental study of Dreze *et al.* [6] which concluded that the level of shelf on which items are displayed within multi-level shelves has an effect on sales and eye-level position is the best. Therefore, the demand rate is assumed to be a function of the displayed inventory level and the display location within shelves of each brand of items. For the location effects, the models of Corstjens and Doyle [4] are considered in the development of the demand function.

Suppose that there are M shelves in a particular categorized area of the store and N brands of items within the category are displayed on these shelves. Due to the full-shelf merchandising policy, the display area is always kept fully stocked and so the displayed inventory of item i always equals X_i. Thus the demand rate of item i can be expressed as

$$D_i = d_i X_i^{\beta_i} \left[\prod_{k \neq i}^{N} X_k^{\beta_{ik}} \right] \cdot a_i \cdot \tag{1}$$

Where $X_i = \sum_{j=1}^{M} X_{ij}$ and $a_i = \dfrac{\sum_{j=1}^{M} X_{ij}\alpha_j}{X_i}$

Now, we consider a deterministic, continuous-review model of an inventory system with the demand rate presented in (1) assuming that $I_i(t)$ is the instantaneous inventory level of item i at time t which includes both the backroom storage and the displayed inventory. With this system, item i is continuously restocked from backroom into shelves as soon as items are sold, so $I_i(t)$ will decrease linearly following the demand rate. Since we assume that the shelves are always kept fully stocked, orders have to be replenished before the backroom inventory reaches zero; that is, at the end of cycle $I_i(T_i)$ equals the number of units of item i displayed on shelves, X_i. Silver and Peterson [13] showed that in the case of a deterministic, constant demand rate and instantaneous replenishment, it is optimal to let the inventory level reaches zero before reordering. The same logic can be extended to this inventory system of item i, so it will be reasonable to replenish orders whenever the backroom inventory reaches zero. The decreasing rate of inventory level of item i at time t is

$$\frac{\partial I_i(t)}{\partial t} = -D_i \tag{2}$$

With the boundary conditions of $I_i(T_i) = X_i$ and $I_i(0)= X_i + Q_i$, we have

$$I_i(t) = (X_i + Q_i) - D_i t, \ 0 \le t \le T_i. \tag{3}$$

From equation (3), the cycle time and average inventory level of item i is

$$T_i = \frac{Q_i}{D_i} \tag{4}$$

$$\bar{I}_i = \frac{(2X_i + Q_i)}{2} \tag{5}$$

From Equations (1), (4) and (5), the retailer's total profit for N brands of items during unit period can be expressed as

$$TP = \sum_{i=1}^{N} [d_i \left\{ (P_i - Cp_i) - \frac{Co_i}{Q_i} \right\} \cdot \prod_{k=1}^{N} X_k^{\beta_{ik}} \cdot a_i - \frac{Ch_i Q_i}{2} - (Ch_i + Cs_i l_i) X_i]. \tag{6}$$

where $X_i = \sum_{j=1}^{M} X_{ij}$ and $a_i = \dfrac{\sum_{j=1}^{M} X_{ij}\alpha_j}{X_i}$ $\forall i$

Given the objective function, the model can be formulated as follows.

[P] $\underset{Q_i, X_{ij}}{Max}$ (6)

s.t

$$\sum_{i=1}^{N} l_i Q_i \le L_B \tag{7}$$

$$\sum_{i=1}^{N} l_i X_{ij} \le L_j \quad \forall j \tag{8}$$

$$X_i^{\min} \le X_i \le X_i^{\max} \quad \forall i \tag{9}$$

$$Q_i^{\min} \le Q_i \le Q_i^{\max} \quad \forall i \tag{10}$$

$$X_{ij} \geq 0 \quad (11)$$
$$\forall i,j$$
$$Q_i \geq 0 \quad (12)$$
$$\forall i$$
,where
$$X_i = \sum_{j=1}^{M} X_{ij}$$
$$\forall i$$
$$a_i = \frac{\sum_{j=1}^{M} X_{ij}\alpha_j}{\sum_{j=1}^{M} X_{ij}}$$
$$\forall i$$

Decision variables are Q_i and X_{ij}, i=1,..., N, j= 1,..., M i.e., order quantity and shelf space allocations of each item, with the objective of maximizing (6) with the constraints of (7) ~ (12). Equation (7) restricts the sum of order quantities to the space of backroom. As a result of (8), the sum of space allocated does not exceed the space for each shelf. Equation (9) provides operational constraint that may be imposed by a retailer to ensure desired minimum and maximum shelf space allocation for each item. And (10) provides operational constraint which may be required from supplier to ensure desired minimum and maximum order quantity for each item

SOLUTION PROCEDURE

[P] is a non-linear programming problem and it is very difficult to find an optimal solution in a closed form. Thus, we develop a solution procedure based on the gradient search. In other words, we attempt to develop an iterative procedure through which the total profit increases. Note that if the values of X_i and Q_i for all i are given, optimal solution X_{ij}^* can be obtained by solving the following linear programming.

[P']
$$\underset{X_{ij}}{Max}\ TP = \sum_{i=1}^{N}\sum_{j=1}^{M} C1_i\,\alpha_j X_{ij} - \sum_{i=1}^{N} C2_i \quad (13)$$
$$\text{s.t.} \quad \sum_{i=1}^{N} l_i X_{ij} \leq L_j \qquad \forall j$$
$$X_i = \sum_{j=1}^{M} X_{ij} \qquad \forall i$$
$$X_{ij} \geq 0 \qquad \forall i,j$$
,where
$$C1_i = d_i\left\{(P_i - Cp_i) - \frac{Co_i}{Q_i}\right\}\cdot\left(\prod_{z\neq i}^{N} X_z^{\beta_{iz}}\right)\cdot X_i^{(\beta_i - 1)} \qquad \forall i$$
$$C2_i = (Ch_i + Cs_i l_i)X_i + Ch_i Q_i/2 \qquad \forall i$$

Therefore, we proceed from the set of X_i and Q_i to the next in such a way that the total profit increases.

The partial derivative of the objective function with respect to X_{ij} is

$$\frac{\partial TP}{\partial X_{ij}} = \sum_{y\neq i}^{N}\left[\begin{array}{l}\beta_{yi}\, d_y\left\{(P_y - Cp_y) - \dfrac{Co_y}{Q_y}\right\} \\ \cdot\left(\prod_{k\neq i} X_k^{\beta_{yk}}\right)\cdot X_i^{(\beta_{yi}-1)}\cdot a_y\end{array}\right] + d_i\left\{(P_i - Cp_i) - \frac{Co_i}{Q_i}\right\}\cdot\left(\prod_{k\neq i} X_k^{\beta_{ik}}\right)\cdot X_i^{(\beta_i - 1)}\cdot\left\{\alpha_j - (1-\beta_i)\cdot a_i\right\} - (Ch_i + Cs_i l_i) \quad (14)$$

Equation (14) represents the ratio of change in the unconstrained profit function to change in X_{ij}. It gives the change in profit that is incurred by increasing X_i if an increment in X_i can be accommodated in shelf j for any given solution. Following Wilson [16], relocations of some items must be considered since the required shelf space may not be available in shelf j when an increment in X_i is accommodated in shelf j. Thus, the gradient of TP^* at X_i is

$G_i^X = \dfrac{\partial TP^*}{\partial X_i}$ = [Rate of change in total profit with respect to X_{ij} for given j] - [Cost incurred by relocation of items, which results from increasing X_i]

The gradient of TP^* at Q_i is

$G_i^Q = \dfrac{\partial TP^*}{\partial Q_i}$ = [Rate of change in total profit with respect to Q_i]

TP^* is an unconstrained function of X_i and Q_i. Since constraint (7) ~ (12) can be satisfied through the following solution procedure, we can proceed from the set of X_i and Q_i to the next in the direction of the positive gradient of unconstrained profit function, TP^*. A solution procedure is developed based on the gradient search (omitted here).

NUMERICAL EXAMPLES

The validity of the model is illustrated with small size problems and solutions are compared with those obtained from a total enumeration. Three data sets are generated for 3 brands of items having negative values of β_{ik}, i.e., substituted items each other. The item parameters are obtained from:

Cp_i=U(2,4), P_i=1.8×Cp_i, Ch_i=0.25×Cp_i, Cs_i= 0.5, Co_i= 5.0, d_i=100/$Cp_i^{0.4}$, l_i=1.0 β_i=U(0.2,0.5), β_{ik}=U(-0.015, -0.005), i = 1, ..., N, $k \neq i$.

$X_i^{\min}=1$, $X_i^{\max} = 1.5\times M$, $Q_i^{\min} = 20$, $Q_i^{\max} = 50$, $i = 1,..N$.

Furthermore, the shelf parameters are given as; L_j= 3, L_B= 100, α_j=1+(M-j) × 0.1, j=1,..., M.

For 3 brands of items and three kinds of shelve levels, total 9 cases are solved both by the proposed solution procedure and by a total enumeration.

Table 1 Computation results with location effects

Number of items	Level of shelf	Data set	Total profit ($) Gradient Search (a)	Total profit ($) Total Enumeration (b)	% Deviation
3	4	1	808.36	823.44	1.831
		2	750.71	759.18	1.116
		3	727.54	736.2	1.176
	5	1	932.65	945.7	1.380
		2	837.34	860.29	**2.668**
		3	817.03	829.14	1.461
	6	1	1059.32	1073.7	1.342
		2	955.29	977.06	2.228
		3	920.37	929.18	**0.948**

* % Deviation = $\frac{|b-a|}{b} \times 100$ (%)

Table 2 Computation results with no location effects

Number of items	Level of shelf	Data set	Total profit ($) Gradient Search (a)	Total profit ($) Total Enumeration (b)	% Deviation
3	4	1	779.22	809.05	3.687
		2	741.21	745.81	0.617
		3	701.51	724.02	3.109
	5	1	906.62	919.91	1.445
		2	822.45	837.37	1.782
		3	797.74	816.15	2.256
	6	1	1009.18	1041.80	3.131
		2	941.21	946.18	0.525
		3	881.06	909.81	3.160

* % Deviation = $\frac{|b-a|}{b} \times 100$ (%)

The values of X_i^* and Q_i^* obtained by the proposed solution procedure are rounded to the nearest integer. We adjust the values of X_i^* and Q_i^* if they don't satisfy the total shelf space and backroom space capacity. [P′] is solved for the rounded values for each set of the data and the solutions are listed in Table 1. As observed in the table, the maximum % deviation is 2.668, minimum % deviation 0.948 and average % deviation 1.572 for all 9 experiments, which indicate that the performance of the proposed solution is in general satisfactory for small sized problems. An interesting question is 'how much effects does the shelf location have on the retailer's total profit?'. To answer the above question, the same problems are solved with α_j being equal to the average of α_j of the original problems. Table 2 shows the computational results. We observe that the total profits become relatively smaller compared with the results in Table 1. Thus we can conclude that from the retailer's point of view, it is more profitable to consider both the shelf location and shelf space effects in solving the shelf space allocation problem. A summary of the test results is shown in Table 3 along with the required computation time. It is observed that % deviation increases as the number of shelf level increases. The computation times required of the proposed solution procedure are much shorter than those of the total enumeration. To confirm that the gradient search procedure is also applicable to large size problem, a problem is solved with 12 brands of items and 6-level self. For the problem, two cases are considered, one with substitute items only, and the other with both substitute and complementary items (the results omitted here).

Table 3 Summary of the test results and computation times

Number of items	Level of shelf	Average % Deviation	Average Value	Average running time (sec.) Gradient Search	Average running time (sec.) Total Enumeration
3	4	1.374		0.043	2153
	5	1.836	1.572	0.059	3810
	6	1.506		0.09	4737

REFERENCES

[1] Anderson, E. E. and Amato, H. N., 1974, A mathematical model for simultaneously determining the optimal brand-collection and display-area allocation, *Operations Research*, 22(1), 13-21.

[2] Borin, N., Farris, P. W. and Freeland, J. R., 1994, A model for determining retail product category assortment and shelf space allocation, *Decision Science*, 25(3), 359-384.

[3] Bultez, A. and Naert, P., 1988, SH.A.R.P: Shelf allocation for retailer's profit, *Marketing Science*, 7(3), 211-231.

[4] Corstjens, M. and Doyle, P., 1981, A model for optimizing retail space allocations, *Management Science*, 27(7), 822-833.

[5] Corstjens, M. and Doyle, P., 1983, A dynamic model for strategically allocating retail space, *Journal of the Operational Research Society*, 34(10), 943-951.

[6] Dreze, X., Hoch, S. J. and Purk, M. E., 1994, Shelf management and space elasticity, *Journal of Retailing*, 70(4), 301-326.

[7] Hansen, P. and Heinsbroek, H., 1979, Product selection and space allocation in supermarkets, *European Journal of Operational Research*, 3(6), 58-63.

[8] Larson, P. D. and DeMarais, R. A., 1990, Psychic stock: an

independent variable category of inventory, *International Journal of Physical Distribution and Logistics Management*, 20(7), 28-34.

[9] Yang, M. H. and Chen, W. C., 1999, A study on shelf space allocation and management, *International Journal of Production Economics*, 60-61, 309-317.