

THE EXTENSION OF ROWE'S STRESS-DILATANCY MODEL TO GENERAL STRESS CONDITION

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ABSTRACT

In this paper, Rowe's stress-dilatancy formulation, originally developed for 2D and triaxial stress conditions, is extended to a more general three-dimensional stress condition. By introducing a second order tensor π defined as a tensor product of stress and strain increment tensors, Rowe's model is re-interpreted in terms of the invariants of this tensor and is expressed in a tensorial form applicable to both 2D and 3D stress conditions. Comparisons with experimental data show that the extended 3D stress-dilatancy formulation can reproduce experimental dilatancy characteristics for general stress conditions.

Key words: 3D stress condition, deformation, dilatancy, granular material (IGC: D6)

INTRODUCTION

The volume change response of granular materials due to shearing was first studied in a qualitative manner by Hagen (1852) and Reynolds (1885). This distinctive feature of granular materials, coined as stress-dilatancy, was attributed to geometrical constraints imposed by the packing of grains as an assembly of particles is sheared. The importance of dilatancy was not fully recognized in the field of soil mechanics until in the 1930's when Casagrande explained the influence of volume changes on the measured friction angle of a soil (Casagrande, 1936, 1938). Following the efforts of Taylor (1948) and Newland and Allely (1957), Rowe (1962) further studied the physical manifestation of dilatancy in two dimensional and axisymmetric three dimensional (i.e. triaxial) stress conditions, and developed a stress-dilatancy model within the framework of modern mechanics.

In Rowe's pioneering work, the stress-dilatancy relation was derived on the basis of simple assembly characteristics taking into account the microscopic deformation mechanism and the hypothesis of minimum energy ratio. Later on, the principles involved in Rowe's stress-dilatancy theory were more rigorously treated by Horne (1965), who also provided a conceptual method to describe the anisotropy of granular materials and found some very important facts about the evolution of fabric anisotropy with respect to dilatancy, such as the existence of a limiting anisotropic state, which can be approached after large deformations have occurred.

Stress-dilatancy theories have been developed either based on energy principles (Newland and Alley, 1957;

Rowe, 1962; Roscoe and Schofield, 1963; Schofield and Wroth, 1968; Moroto, 1976; Dafalias, 1987; Muhunthan, Chemeau and Masad, 1996) or on the kinematics of particles by considering the constraint imposed by internal grain geometry during macroscopic deformations (Matsuoka, 1974; Tokue, 1979a; Goddard and Bashir, 1990). Within the realm of constitutive modelling, stress-dilatancy equations used in conjunction with an elasto-plastic model are abundant in the literature, e.g. Nova and Wood (1979), Matsuoka (1974), Wan and Guo (1998, 1999) and Li (2002). There have also been various conceptual contributions based on micro-mechanics (Nemat-Nasser and Mehrabadi, 1983; Oda, 1972, 1974; Matsuoka and Takeda, 1980; Guo, 2000; Wan and Guo, 2001a, b) to describe the effect of fabric on stress-dilatancy equations. However, these studies introduce various assumptions that are difficult to verify experimentally. Consequently, by far, the most popular descriptions of stress-dilatancy still hinge on energy concepts as in Rowe's pioneering work (Rowe, 1962).

For a regular packing of spheres or cylinders as an analog to granular materials, when assuming a purely frictional deformation mechanism, all particles would slip relative to each other at contacts on different local shearing planes and eventually move in a fixed mean direction with an angle from the macroscopic or average shearing plane. The resulting stress-dilatancy equation, derived for either plane strain or triaxial test conditions, relates stress ratio to dilatancy. It is commonly used as a flow rule in plasticity theory. However, Rowe's minimum energy ratio hypothesis is difficult to apply to some specific stress paths, for example, one dimensional

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Manuscript was received for review on January 23, 2003

Written discussions on this paper should be submitted before March 1, 2005 to the Japanese Geotechnical Society, 4-38-2, Sengoku, Bunkyo-ku, Tokyo 112-0011, Japan. Upon request the closing date may be extended one month.

compression in which the lateral strain is zero. Furthermore, as reported by Horne (1965) and Tatsuoka (1976), the extension of Rowe's stress-dilatancy formulation to three-dimensional stress conditions is not straightforward. Tokue (1979b) proposed a different approach based on microscopic analysis to interpret the dilatancy behaviour of granular material, however, the results are inconvenient to use. Attempts have been made to reinterpret Rowe's minimum energy ratio hypothesis to general stress conditions by introducing a combined tensor based on the stress and strain tensors; *see*, for example, Frossard (1983) and Satake (1987).

This paper presents an approach for the extension of Rowe's stress-dilatancy model to 3D stress conditions. After introducing a second order tensor π that is defined as a tensor product of stress and strain increment tensors, Rowe's stress-dilatancy equation for 2D and triaxial stress conditions is reinterpreted in terms of the invariants of tensor π and is then expressed in a tensorial function that is applicable to both 2D and 3D stress conditions. Comparisons of predictions and experimental data of true triaxial tests demonstrate that the extended stress-dilatancy formulation can reproduce the salient dilatancy characteristics of granular materials under 3D stress conditions.

RE-INTERPRETATION OF ROWE'S STRESS-DILATANCY EQUATION

Preliminary

Following Frossard (1983) and Satake (1987), for the stress tensor σ_{ij} and the strain increment tensor $d\varepsilon_{ij}$, let us define a tensor π_{ij} as follows

$$\pi_{ij} = \frac{1}{2}(\sigma_{ik}d\varepsilon_{kj} + \sigma_{jk}d\varepsilon_{ki}) \quad (1)$$

When the stress tensor σ_{ij} and the strain increment tensor $d\varepsilon_{ij}$ are coaxial, Eq. (1) reads

$$\pi_{ij} = \sigma_{ik}d\varepsilon_{kj} \quad (2)$$

with the principal components of π_{ij} defined as

$$\pi_1 = \sigma_1 d\varepsilon_1, \quad \pi_2 = \sigma_2 d\varepsilon_2, \quad \pi_3 = \sigma_3 d\varepsilon_3 \quad (3)$$

where σ_1 , σ_2 and σ_3 are the major, the intermediate and the minor principal stresses, $d\varepsilon_1$, $d\varepsilon_2$ and $d\varepsilon_3$ are the corresponding principal strain increments. For 2D (or biaxial) cases where $d\varepsilon_3$ is assumed to be 0, two invariants of π_{ij} can be defined as follows

$$u = \frac{\pi_1 + \pi_2}{2} = \frac{\sigma_1 d\varepsilon_1 + \sigma_2 d\varepsilon_2}{2} \quad (4)$$

$$v = \frac{\pi_1 - \pi_2}{2} = \frac{\sigma_1 d\varepsilon_1 - \sigma_2 d\varepsilon_2}{2} \quad (5)$$

where u and v are spherical and deviatoric invariants of π_{ij} , respectively. On the other hand, by drawing parallels with the stress invariants used for triaxial conditions with $\sigma_1 > \sigma_2 > \sigma_3$, u and v are expressed as

$$u = \frac{\pi_1 + 2\pi_3}{3} = \frac{1}{3}(\sigma_1 d\varepsilon_1 + 2\sigma_3 d\varepsilon_3) \quad (6)$$

$$v = \pi_1 - \pi_3 = \sigma_1 d\varepsilon_1 - \sigma_3 d\varepsilon_3 \quad (7)$$

Herein the sign convention in soil mechanics is taken such that compressive stress and contractive volumetric strain are positive. Given that π_{ij} is a second-order tensor, for more general stress conditions, we can define the spherical and the deviatoric invariants of π_{ij} as

$$u = \frac{1}{3}Tr(\pi_{ij}), \quad v = \left(\frac{3}{2}\pi'_{ij}\pi'_{ij}\right)^{1/2} \quad (8)$$

with π'_{ij} being the deviator of tensor π_{ij} in the form of

$$\pi'_{ij} = \pi_{ij} - \frac{1}{3}\pi_{kk}\delta_{ij} \quad (9)$$

One can also define the second and the third invariants of π'_{ij} as

$$J_{2\pi} = \frac{1}{2}\pi'_{ij}\pi'_{ij}, \quad J_{3\pi} = \frac{1}{3}\pi'_{ik}\pi'_{kj}\pi'_{ij} \quad (10)$$

respectively. One finds that the trace of π_{ij} defined in Eq. (1) represents the work increment (scalar) defined as $dW = \sigma_{ij}d\varepsilon_{ij}$.

Rowe's Stress-Dilatancy Equation for 2D Stress Conditions

Based on a purely frictional deformation mechanism and the assumption that the dilatancy is due to the internal geometry constraint, as well as the hypothesis of minimum energy ratio, which is expressed as $E = (\text{work in})/(\text{work out}) = \text{minimum}$, Rowe (1962) deduced the following relation for two-dimensional stress systems:

$$-\frac{\sigma_1 d\varepsilon_1}{\sigma_2 d\varepsilon_2} = \tan^2\left(\frac{\pi}{4} + \frac{\phi_f}{2}\right) = \frac{1 + \sin \phi_f}{1 - \sin \phi_f} \quad (11)$$

where ϕ_f is a characteristic friction angle and the strain increments $d\varepsilon_1$ and $d\varepsilon_2$ are understood as being plastic, i.e. elastic deformations are ignored. When defining the dilation angle ψ as

$$\sin \psi = -\frac{d\varepsilon_v}{d\gamma} \quad (12)$$

and invoking the mobilized friction angle ϕ_m via Mohr-Coulomb criterion such that $\sin \phi_m = (\sigma_1 - \sigma_2)/(\sigma_1 + \sigma_2)$, Rowe's equation becomes

$$\sin \psi = \frac{\sin \phi_m - \sin \phi_f}{1 - \sin \phi_m \sin \phi_f} \quad (13)$$

The characteristic friction angle ϕ_f is different from the mobilized friction angle ϕ_m . According to Rowe (1971) and de Josselin de Jong (1976), ϕ_f is the friction angle on an instantaneous mean sliding plane along which the net movements of soil particles take place. As demonstrated in Fig. 1, while failure occurring on an inclined saw-tooth surface ab , particles move along different local sliding directions with the mean being AB , which deviates from ab and hence induces volume change. Being defined as the friction angle along the mean direction (AB) of local

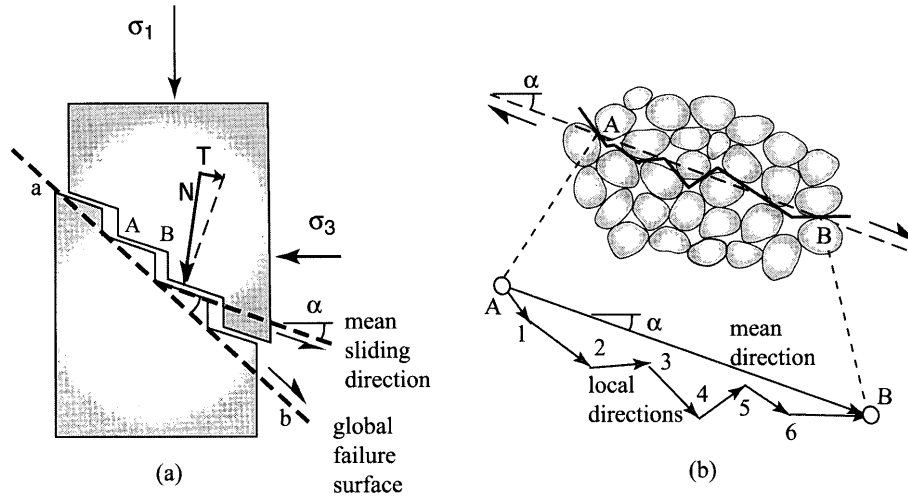


Fig. 1. (a) Inclined saw-tooth failure surface and (b) local and mean interparticle sliding directions

sliding, ϕ_f is influenced by particle arrangements and the number of sliding contacts. According to Mohr-Coulomb criterion, ϕ_m is in fact the mobilized friction angle on the failure surface ab . Consequently, ϕ_f takes the same value of ϕ_m only when the mean sliding direction AB is identical to ab , leading to the rate of volume change being instantaneously zero. In other words, instead of a constant, ϕ_f takes different values for granular materials of different packing of particles at various strain levels (Rowe, 1971; Wan and Guo, 1998). However, ϕ_f is usually considered to be the same as the critical friction angle ϕ_{cv} mainly for the sake of simplicity; see, for example, Vermeer (1982).

Referring to the invariants defined in Eqs. (4) and (5), Rowe's stress-dilatancy relation in Eq. (11) can be expressed as

$$\frac{u}{v} = \frac{\pi_1 + \pi_2}{\pi_1 - \pi_2} = \frac{\sigma_1 d\epsilon_1 + \sigma_2 d\epsilon_2}{\sigma_1 d\epsilon_1 - \sigma_2 d\epsilon_2} = \sin \phi_f \quad (14)$$

which yields

$$(1 - \sin \phi_f) \pi_1 + (1 + \sin \phi_f) \pi_2 = 0 \quad (15)$$

or after defining

$$A_1 = 1 - \sin \phi_f, A_2 = 1 + \sin \phi_f \quad (16)$$

Equation (15) is rewritten as

$$\Pi = A_i \pi_i = 0, A_1 + A_2 = 2 \quad (17)$$

With the assumption of coaxiality between the stress and the strain increment, the above equation can be extended to a more general form

$$\Pi = A_{ij} \pi_{ij} = 0, A_{ii} = \delta_{ii} = 2 \quad (18)$$

Recalling that π_{ij} is a second-order tensor with $\pi_1 > \pi_2$, A_{ij} can also be regarded a second-order tensor in which $A_1 < A_2$.

Rowe's Stress-Dilatancy Equation for Triaxial Stress Conditions

For triaxial compression stress conditions ($\sigma_1 > \sigma_2 = \sigma_3$), Rowe's stress-dilatancy equation based on the

hypothesis of minimum energy ratio reads

$$-\frac{\sigma_1 d\epsilon_1}{2\sigma_3 d\epsilon_3} = \tan^2 \left(\frac{\pi}{4} + \frac{\phi_f}{2} \right) = \frac{1 + \sin \phi_f}{1 - \sin \phi_f} \quad (19)$$

which can be rewritten in the form

$$\frac{3u}{v} = \frac{\sigma_1 d\epsilon_1 + 2\sigma_3 d\epsilon_3}{\sigma_1 d\epsilon_1 - \sigma_3 d\epsilon_3} = \frac{4 \sin \phi_f}{3 + \sin \phi_f} \quad (20)$$

Similar to Eq. (18), the general expression of Rowe's stress-dilatancy relation for triaxial compression conditions may be given by

$$\Pi = A_{ij} \pi_{ij} = 0, A_{ii} = \delta_{ii} = 3 \quad (21)$$

with the principal components of A_{ij} in triaxial compression as

$$A_1 = 1 - 2R_w^{ctc}, A_2 = A_3 = 1 + R_w^{ctc}, R_w^{ctc} = \frac{2 \sin \phi_f}{3 + \sin \phi_f} \quad (22)$$

in which $A_1 < A_2 = A_3$ and ϕ_f is the characteristic friction angle. Recalling that π_{ij} and A_{ij} are both second-order tensors, by observation, it is found that π_{ij} and A_{ij} are non-coaxial and

$$\cos(\pi_{ij}, A_{ij}) = -1 \quad (23)$$

For triaxial extension stress conditions ($\sigma_1 = \sigma_2 > \sigma_3$), Rowe's stress-dilatancy relation takes the following form

$$-\frac{2\sigma_1 d\epsilon_1}{\sigma_3 d\epsilon_3} = \tan^2 \left(\frac{\pi}{4} + \frac{\phi_f}{2} \right) = \frac{1 + \sin \phi_f}{1 - \sin \phi_f} \quad (24)$$

or

$$\frac{3u}{v} = \frac{2\sigma_1 d\epsilon_1 + \sigma_3 d\epsilon_3}{\sigma_1 d\epsilon_1 - \sigma_3 d\epsilon_3} = \frac{4 \sin \phi_f}{3 - \sin \phi_f} \quad (25)$$

which with the help of Eq. (21) leads to

$$A_1 = A_2 = 1 - R_w^{cte}, A_3 = 1 + 2R_w^{cte}, R_w^{cte} = \frac{2 \sin \phi_f}{3 - \sin \phi_f} \quad (26)$$

In this case, Eq. (23) is still satisfied since $A_1 = A_2 < A_3$ and $\sigma_1 = \sigma_2 > \sigma_3$.

At this point, it should be observed that the results

presented in Eqs. (14), (20) and (25) show that Rowe's hypothesis of minimum energy ratio for 2D and triaxial stress conditions can be reinterpreted in terms of the invariants of tensor π_{ij} . More generally, Rowe's relation can be expressed as functions of π_{ij} by Eqs. (18) and (21), which are convenient for extending Rowe's stress-dilatancy theory to three-dimensional stress conditions. In fact, Eq. (21) can be regarded as the general expression of Rowe's stress-dilatancy formulation for 3D stress conditions; however, the second-order tensor A_{ij} , associated with the characteristic friction angle ϕ_f , remains to be determined.

ROWE'S STRESS-DILATANCY FORMULATION IN 3D

Extension of Rowe's Hypothesis of Minimum Energy Ratio

According to the previous section, the 3D extension of the hypothesis of minimum energy ratio can be expressed as

$$\Pi = A_{ij}\pi_{ij} = 0, \quad A_{ii} = \delta_{ii} = 3, \quad \cos(\pi_{ij}, A_{ij}) = -1 \quad (27)$$

Since A_{ij} and π_{ij} are second-order tensors, $\Pi = A_{ij}\pi_{ij}$ can be expressed as

$$\Pi = A_s u + |A_{ij}^d| |\pi_{ij}^d| \cos(\pi_{ij}^d, A_{ij}^d) = A_s u - |A_{ij}^d| |\pi_{ij}^d| \quad (28)$$

where

$$A_s = \frac{1}{3}(A_1 + A_2 + A_3) = 1 \quad (29)$$

is the spherical component of A_{ij} , and

$$|A_{ij}^d| = (A_{ij}^d A_{ij}^d)^{1/2}, \quad A_{ij}^d = A_{ij} - \delta_{ij} \quad (30)$$

with $|A_{ij}^d|$ being the modulus of A_{ij}^d that is the deviator of A_{ij} . If the principal components of A_{ij} and π_{ij} are used, Eq. (27) simplifies to

$$\Pi = A_i \pi_i \quad (31)$$

where π_i is defined by Eq. (3). In order to provide an explicit criterion for the 3D extension of the hypothesis of minimum energy ratio expressed in Eq. (27), A_i is chosen as

$$A_i = \frac{3\hat{I}_3}{\hat{I}_2} \frac{1}{\hat{\sigma}_i} \quad (32)$$

where $\hat{\sigma}$ is the stress tensor at a certain stress state, \hat{I}_2 and \hat{I}_3 are the second and the third invariants of $\hat{\sigma}$ respectively. Given Eq. (3) and substituting Eq. (32) into Eq. (31), one has

$$\Pi = A_i \pi_i = \frac{3\hat{I}_3}{\hat{I}_2} d\varepsilon_v \quad (33)$$

with $d\varepsilon_v = d\varepsilon_{ii}$ being the volumetric strain. If $\hat{\sigma}$ is chosen as the stress at the characteristic state when $d\varepsilon_v = 0$ similar to that for 2D and triaxial stress conditions, then Eq. (27) is satisfied. After some algebraic manipulations, it can be shown that the modulus of A_{ij}^d may be expressed as

$$|A_{ij}^d| = (A_{ij}^d A_{ij}^d)^{1/2} = \sqrt{6} R_w \quad (34)$$

with

$$R_w = \left(1 - 3 \frac{\hat{I}_1 \hat{I}_3}{\hat{I}_2^2}\right)^{1/2} \quad (35)$$

in which $\hat{I}_1 = \hat{\sigma}_{ii}$. Inserting Eqs. (29) and (34) into Eq. (28) immediately yields

$$3u - \sqrt{6} R_w |\pi_{ij}^d| = 0 \quad (36)$$

or

$$\frac{3u}{v} = 2R_w = 2 \left(1 - 3 \frac{\hat{I}_1 \hat{I}_3}{\hat{I}_2^2}\right)^{1/2} \quad (37)$$

where $3u = Tr(\pi_{ij}) = dW$ is the work increment and $v = (3J_{2\pi})^{1/2}$ with $J_{2\pi}$ being the second deviatoric invariant of π_{ij} . For conventional triaxial compression and extension conditions, R_w in Eq. (35) takes the simple forms

$$R_w^{ctc} = \sqrt{1 - 3 \frac{R(R+2)}{(2R+1)^2}} = \frac{2 \sin \phi_f}{3 + \sin \phi_f} \quad (38)$$

$$R_w^{cte} = \sqrt{1 - 3 \frac{2R+1}{(R+2)^2}} = \frac{2 \sin \phi_f}{3 - \sin \phi_f} \quad (39)$$

with $R = \hat{\sigma}_1/\hat{\sigma}_3 = \tan^2(\pi/4 + \phi_f/2)$, which confirms that the expressions of R_w^{ctc} and R_w^{cte} in Eqs. (22) and (26) are recovered.

Equation (37) is the expression of Rowe's hypothesis of minimum energy ratio for three dimensional stresses, which relates the ratio of the work increment to the deviator of π_{ij} with the stresses of a characteristic state at which the rate of volume change is instantaneously zero. Providing the explicit expression of u and v together with the invariants of π_{ij} , Rowe's stress-dilatancy relation for three dimensional stresses can be readily extended.

Extension of Rowe's Stress-Dilatancy Relation to General Stress Conditions

Given the stress and the strain increment, the work increment can be obtained from $dW = Tr(\pi_{ij}) = \sigma_{ij} d\varepsilon_{ij}$, i.e.

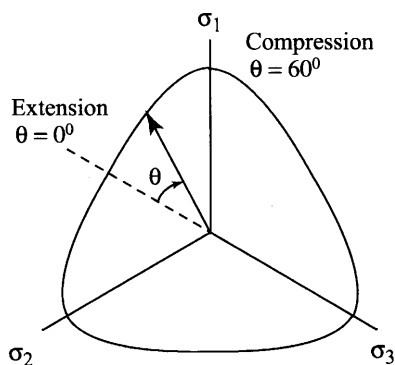
$$dW = 3u = \frac{I_1}{3} d\varepsilon_v + |s| |d\mathbf{e}| \cos \delta_\alpha \quad (40)$$

where $|\bullet|$ denotes the modulus of a tensor, δ_α is the non-coaxiality angle between the stress and the strain increment tensors in the deviatoric plane, s and $d\mathbf{e}$ are the deviatoric stress and the deviatoric strain increment tensor respectively. After some tensorial manipulations, the deviatoric invariant of π_{ij} is given as

$$v = (3J_{2\pi})^{1/2} = \sqrt{\frac{3}{2}} \left[B + \left(\frac{I_1}{3} |d\mathbf{e}| + \frac{1}{3} |s| d\varepsilon_v \right)^2 \right]^{1/2} \quad (41)$$

with

$$B = \left(m_{ik} n_{kj} m_{ij} n_{il} - \frac{2 \cos \delta_\alpha - 1}{3} \right) (|s| |d\mathbf{e}|)^2 + \left[\frac{2}{3} (I_1 |d\mathbf{e}|) m_{ik} n_{kj} n_{ij} + \frac{2}{3} (|s| d\varepsilon_v) m_{ik} n_{kj} m_{ij} \right] |s| |d\mathbf{e}| \quad (42)$$

Fig. 2. Lode's angle on the π -plane

in which \mathbf{m} and \mathbf{n} are defined as $\mathbf{m} = \mathbf{s}/|\mathbf{s}|$ and $\mathbf{n} = d\mathbf{e}/|d\mathbf{e}|$ with $\cos \delta_\alpha = m_{ij}n_{ij}$. Obviously, ν is a function of the non-coaxiality angle δ_α . When the stress and the strain increment tensors are coaxial, i.e. $\delta_\alpha = 0$, Eq. (42) is simplified as

$$B = \frac{1}{6} \left[(|\mathbf{s}| |d\mathbf{e}|) + 4\sqrt{3} \left(\frac{I_1}{3} |d\mathbf{e}| + \frac{1}{3} |\mathbf{s}| d\epsilon_v \right) \cos 3\theta \right] |\mathbf{s}| |d\mathbf{e}| \quad (43)$$

where θ is Lode's angle of stress tensor σ , as shown in Fig. 2. The following section will focus on the cases in which the stress and the strain increment tensors are coaxial.

Following the conventions of soil mechanics by introducing

$$d\gamma = \left(\frac{3}{2} d\epsilon_{ij} d\epsilon_{ij} \right)^{1/2} = \sqrt{\frac{3}{2}} |d\mathbf{e}|, \quad (44)$$

$$q = \left(\frac{3}{2} s_{ij} s_{ij} \right)^{1/2} = \sqrt{\frac{3}{2}} |\mathbf{s}|, \quad p = \frac{I_1}{3}$$

with $d\gamma$ being the equivalent shear strain increment, q the deviatoric stress and p the mean effective stress, the expressions of u and ν become

$$3u = pd\epsilon_v + \frac{2}{3} qd\gamma, \quad \nu = \left(pd\gamma + \frac{1}{3} qd\epsilon_v \right) (1+X)^{1/2} \quad (45)$$

with

$$X = \frac{1}{9} \eta^2 - \frac{2}{3} \eta \cos 3\theta \quad (46)$$

and $\eta = q/p$. Finally, the 3D expression of Rowe's hypothesis of minimum energy ratio given in Eq. (37) becomes

$$\frac{pd\epsilon_v + \frac{2}{3} qd\gamma}{pd\gamma + \frac{1}{3} qd\epsilon_v} = 2(1+X)^{1/2} R_w \quad (47)$$

which after reorganizing yields

$$-\frac{d\epsilon_v}{d\gamma} = \frac{\frac{2}{3} \eta - 2(1+X)^{1/2} R_w}{1 - \frac{2}{3} \eta (1+X)^{1/2} R_w} \quad (48)$$

or

$$-\frac{d\epsilon_v}{d\gamma} = \frac{2}{3} \frac{\eta - \eta_f}{1 - \frac{2}{9} \eta \eta_f}; \quad \eta_f = 3(1+X)^{1/2} R_w \quad (49)$$

where η_f is the stress ratio at characteristic state, while R_w and X are given in Eqs. (35) and (46), respectively. For triaxial stress conditions, by relating the stress ratios η_f and η to ϕ_f and the mobilized friction angle ϕ_m respectively, it follows that

$$-\frac{d\epsilon_v}{d\gamma} \approx \frac{4}{3} \frac{\sin \phi_m - \sin \phi_f}{1 - \sin \phi_m \sin \phi_f} \quad (50)$$

with $d\gamma = d\epsilon_1 - d\epsilon_3$. Equation (50) recovers the original expression of stress-dilatancy relation for triaxial stress conditions; see, for example, Vermeer (1982).

Physical Meaning of the Deviatoric Invariant of π

For simplicity, let us consider 2D stress conditions in which the deviatoric components of tensor π are

$$\pi'_{11} = \frac{1}{2} (\sigma_{11} d\epsilon_{11} - \sigma_{22} d\epsilon_{22}); \quad \pi'_{22} = -\frac{1}{2} (\sigma_{11} d\epsilon_{11} - \sigma_{22} d\epsilon_{22}) \quad (51)$$

$$\pi'_{12} = \frac{1}{2} (\sigma_{11} d\epsilon_{12} + \sigma_{22} d\epsilon_{21} + \sigma_{12} d\epsilon_{22} + \sigma_{21} d\epsilon_{11}) \quad (52)$$

Using the invariants of stress and strain increment tensors, ν can be expressed as

$$\nu = \left(\frac{1}{2} \pi'_{ij} \pi'_{ij} \right)^{1/2} = \frac{1}{2} [(pd\gamma)^2 + (qd\epsilon_v)^2 + 2(pd\gamma)(qd\epsilon_v) \cos 2\delta_\alpha]^{1/2} \quad (53)$$

with $d\gamma = \sqrt{2d\epsilon_{ij}d\epsilon_{ij}}$, $q = \sqrt{s_{ij}s_{ij}/2}$ and δ_α being the non-coaxiality angle between the stress and the strain increment tensors. When $\delta_\alpha = 0$, Eq. (53) becomes

$$\nu = \frac{pd\gamma + qd\epsilon_v}{2} \quad (54)$$

From Eqs. (53) and (54), one observes that the quantity ν is a special work increment done by unusual combinations of $pd\gamma$ and $qd\epsilon_v$. Moreover, ν also reflects the influence of non-coaxiality between the stress and strain increment tensors. When the angle of dilation $\psi = 0$ (i.e., the shear induced volume change $d\epsilon_v = 0$), one has

$$\nu_0 = \frac{1}{2} pd\gamma \quad (55)$$

which results in

$$\nu = \nu_0 \left[1 + \left(\frac{q}{p} \frac{d\epsilon_v}{d\gamma} \right)^2 + 2 \frac{q}{p} \frac{d\epsilon_v}{d\gamma} \cos 2\delta_\alpha \right]^{1/2} \quad (56)$$

one finds that ν might be interpreted as a special term of work increment that reflects the effect of dilatancy. For a particular case when the stress and strain increment tensors are coaxial and $d\epsilon_v/d\gamma = 1$, ν becomes

$$\nu = \frac{pd\epsilon_v + qd\gamma}{2} = \frac{1}{2} dW \quad (57)$$

Equation (57) indicates that ν can be regarded as a quan-

tity describing the work increment (scalar) defined by $dW = \sigma_{ij} d\epsilon_{ij}$ when the dilation angle defined in Eq. (12) equals to $-\pi/2$. More detailed study is required to obtain deeper insight into the physical meaning of the deviator of tensor π_{ij} .

Characteristic Stress State Surface

With the characteristic state being defined as the state at which the rate of volume change is instantaneously zero, it can be expressed for three dimensional stresses as a surface on the π -plane given by

$$\eta = \eta_f = 3(1 + X)^{1/2} R_w \tag{58}$$

The expression of R_w is determined as follows. Following the discussion about ϕ_f in Rowe's Stress-Dilatancy Equation for 2D Stress Conditions, as the first order of approximation, η_f is considered being the same as η_{cr} at the critical state when $d\epsilon_v/dy = 0$. Given Eq. (48), one has the stress ratio at critical state

$$\eta_{cr} = 3(1 + X_{cr})^{1/2} R_w \tag{59}$$

with

$$X_{cr} = \frac{1}{9} \eta_{cr}^2 - \frac{2}{3} \eta_{cr} \cos 3\theta \tag{60}$$

Consequently, R_w is expressed as

$$R_w = \frac{1}{3} \frac{\eta_{cr}}{(1 + X_{cr})^{1/2}} = \frac{1}{3} \frac{\eta_{cr}}{\left(1 + \frac{1}{9} \eta_{cr}^2 - \frac{2}{3} \eta_{cr} \cos 3\theta\right)^{1/2}} \tag{61}$$

and the stress-dilatancy relation for three-dimensional stress given in Eq. (49) becomes

$$-\frac{d\epsilon_v}{dy} = \frac{2}{3} \frac{\eta - \eta_f}{1 - \frac{2}{9} \eta \eta_f}; \quad \eta_f = \eta_{cr} F(\theta) \tag{62}$$

in which

$$F(\theta) = \left(\frac{1 + \frac{1}{9} \eta^2 - \frac{2}{3} \eta \cos 3\theta}{1 + \frac{1}{9} \eta_{cr}^2 - \frac{2}{3} \eta_{cr} \cos 3\theta} \right)^{1/2} \tag{63}$$

describes the shape of the characteristic state surface that is a function of Lode's angle θ and the current deviatoric stress ratio η . At critical deformation states, the stresses usually satisfy Matsuoka-Nakai criterion (Matsuoka and Nakai, 1982)

$$\frac{I_1 I_2}{I_3} = 9 + 8 \tan^2 \phi_{cv}^{ctc} \tag{64}$$

in which ϕ_{cv}^{ctc} is the friction angle at critical state under triaxial compression conditions. In terms of stress ratio η and the Lode's angle θ , Eq. (64) can be expressed as

$$\eta = \frac{3}{\sqrt{2}} C_1 g(\theta) \tag{65}$$

with

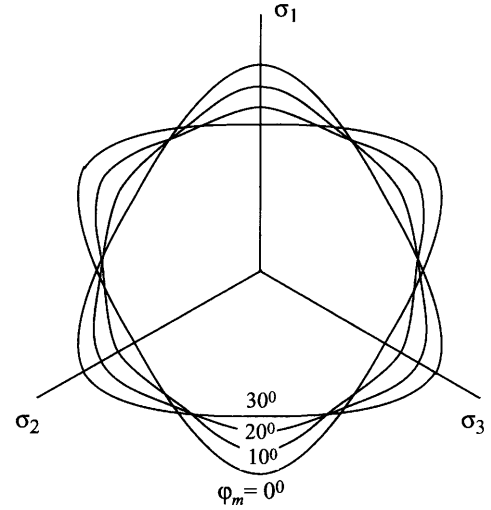


Fig. 3. The dependency of characteristic state surface on current stress ratios

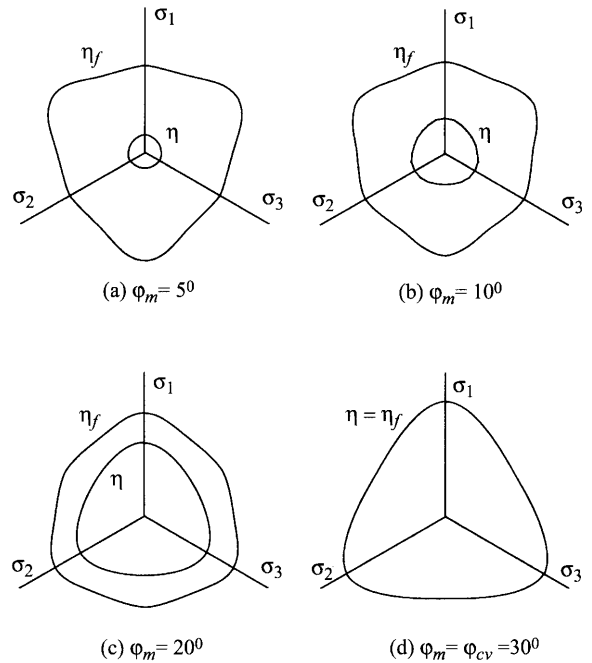


Fig. 4. The evolution of characteristic state surfaces and current loading surfaces

$$g(\theta) = \frac{1}{\left(\cos \frac{1}{3} \cos^{-1}(\chi \cos 3\theta)\right)}, \quad C_1 = \sqrt{\frac{2 \tan^2 \phi_{cv}^{ctc}}{3 + 4 \tan^2 \phi_{cv}^{ctc}}}$$

$$\chi = \frac{(3 - 2C_1)C_1}{\sqrt{2}} \tag{66}$$

Figure 3 provides examples of η_f at different mobilized friction angles with $\phi_{cv} = 30^\circ$. The shape of the initial characteristic state surface at $\eta = 0$ is very close to that of Matsuoka-Nakai model at critical state, however, rotated by $\pi/3$. With an increase of η (or ϕ_m), one observes that the shape of the characteristic state surface changes gradually until it finally coincides with Matsuoka-Nakai

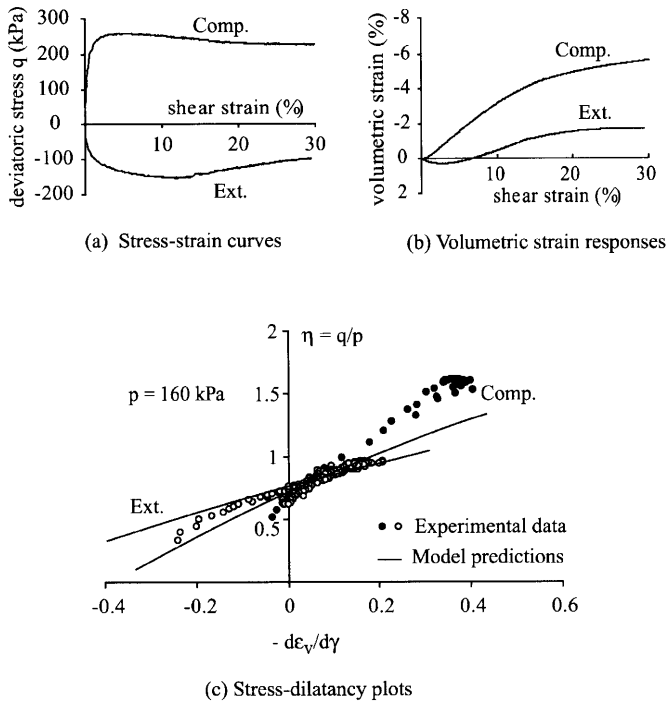


Fig. 5. Stress strain responses and dilatancy plots in triaxial compression and extension (Data after Velacs Project)

model when the critical state at which point $d\varepsilon_v/d\gamma=0$ is achieved.

The 3D stress-dilatancy relation given in Eq. (62) shows that the rate of dilation depends on the difference between current stress ratio η and characteristic stress ratio η_r that is a function of Lode's angle and soil properties at critical state. Figure 4 demonstrates the evolution of the loading surface and the characteristic state surface during a deformation process. One observes that the variation of $\eta - \eta_r$ with the stress ratio η is influenced by Lode's angle, resulting in different dilations when Lode's angle is changed, particularly for the cases of triaxial extension and compression in which Lode's angle θ is 0 and $\pi/3$, respectively.

MODEL PERFORMANCE

Triaxial Stress Conditions

Figures 5(a) and (b) show typical stress and volumetric responses in a series of drained triaxial tests from the Velacs Project (<http://geoinfo.usc.edu/gees/velacs/>). When the mean effective stress is kept constant during deformation, more volumetric compression, both the rate (i.e. $d\varepsilon_v/d\gamma$) and the amount, is observed at small strain in the triaxial extension test. But in contrast, higher dilation develops in triaxial compression condition at large shear strain, inducing higher mobilized stress ratio. The stress-dilatancy relations plotted in Fig. 5(c) show reasonable agreement between experimental data and the extended stress-dilatancy equation. One observes that the stress-dilatancy plot for triaxial compression is steeper, indicating that the rate of dilation $-d\varepsilon_v/d\gamma$ has more

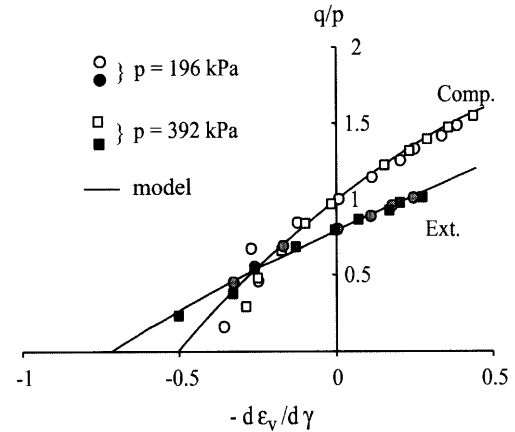


Fig. 6. Stress-dilatancy plots for triaxial compression and extension tests (Data after Nakai, 1987)

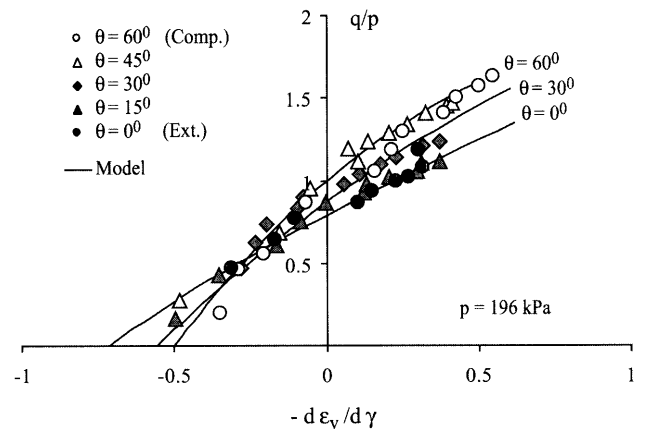


Fig. 7. Effects of Lode's angle on stress-dilatancy plots (Data after Nakai, 1987)

effect on the mobilization of stress ratio $\eta=q/p$. The experimental data of Nakai (1987) also confirm these observations, as shown in Fig. 6. Figures 5(c) and 6 show that the extended 3D stress-dilatancy model reasonably reproduces the dilatancy characteristics under both triaxial compression and extension conditions, including the larger gradient of stress-dilatancy plot for triaxial compression and the higher initial rate of volume compaction in triaxial extension since the initial values of η_r are different for compression and extension conditions; see Fig. 4.

It should be mentioned that the experimental stress-dilatancy plots in Fig. 5(c) show only the pre-peak stress-dilatancy relation and are plotted using total strain increments instead of plastic ones due to insufficient information for elastic deformations. One can argue that the dilatancy plots by considering plastic strain increments will become closer to the model predictions.

General Stress Conditions

Nakai (1987) and Miyamori (1976) conducted true-triaxial tests on sand along stress paths with constant Lode's angle, as demonstrated in Fig. 2. The compari-

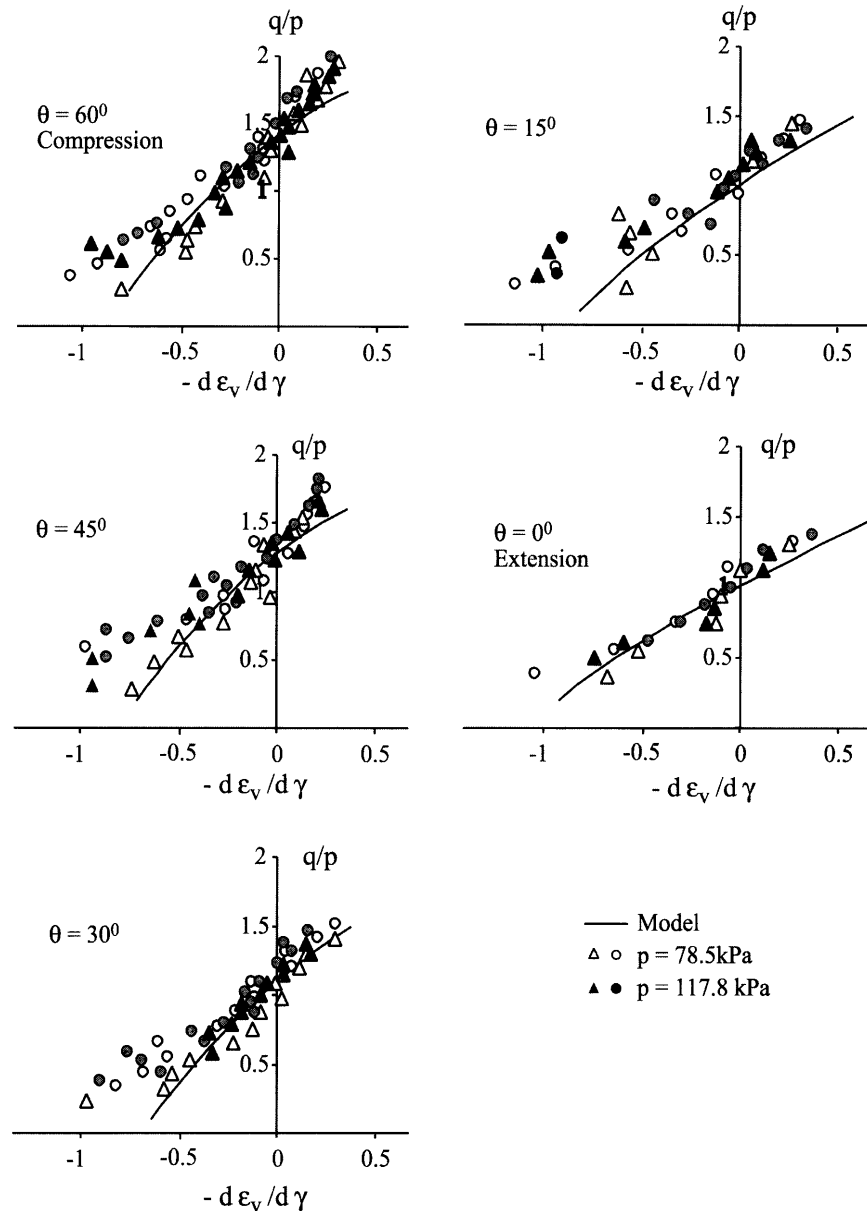


Fig. 8. Effects of Lode's angle on stress-dilatancy plots (Data after Miyamori, 1976)

sons between model predictions and the experimental stress-dilatancy plots with different Lode's angles are shown in Figs. 7 and 8, in which ϕ_{cv} is assumed to be 30° and 35° for the data of Nakai and Miyamori, respectively. The following are noted after an examination of Figs. 7 and 8:

(1) Although the experimental data are scattered, the dilatancy plots shift upward with an increase of θ as defined in Fig. 2;

(2) The gradient of a dilatancy plot increases with θ ; and

(3) The predictions of the extended stress-dilatancy equation for general stress conditions are generally in good agreement with experimental data.

The increase in the gradient of a stress-dilatancy plot with Lode's angle θ indicates the effect of the intermediate principal stress σ_2 on dilatancy. With an increase of θ ,

the intermediate principal stress σ_2 , similar to the major principal stress σ_1 when $\theta=0$ in triaxial extension, decreases gradually until $\sigma_2=\sigma_3$ when $\theta=\pi/3$ at triaxial compression, resulting in a decrease in the confinement to volume change and hence large dilation. This deformation phenomenon also reveals that dilation becomes more important when a granular material is deformed along stress paths with large θ , which provides more kinematical constraint on material deformation.

Based on a series of true triaxial tests using Fuji River and Toyoura sands, Miura and Toki (1984) proposed the following stress-dilatancy relationship

$$-\frac{d\epsilon_v}{d\gamma} = \frac{2}{3} C_d (\eta - m) \quad (67)$$

where C_d is the reciprocal of the linear gradient of a $(-d\epsilon_v/d\gamma) \sim \eta$ line, m is the value of η at the maximum

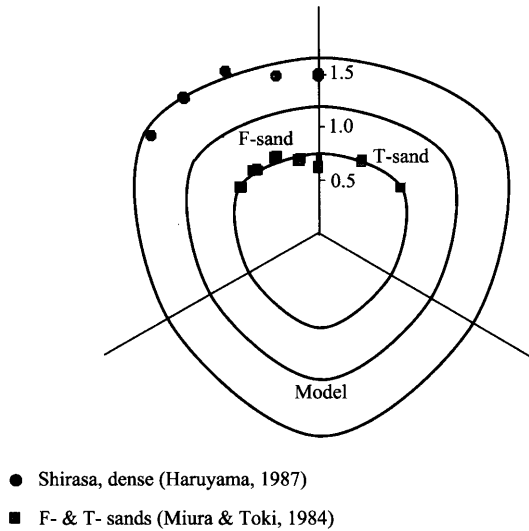


Fig. 9. The reciprocal of stress-dilatancy plots on the π -plane: experimental results and model predictions

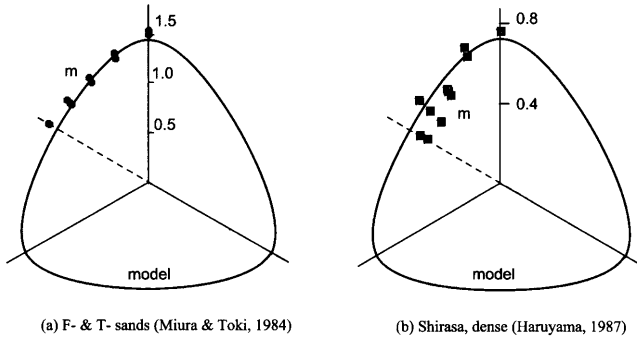


Fig. 10. Characteristic state surface at the maximum volume compression during shearing

volume contraction when the rate of volume is instantaneously zero. On the π -plane, the contour of m represents a surface of characteristic stress states. Equation (67) also fits the experimental data of Haruyama (1987) for Shirasa (a volcanic sandy soil) under three-dimensional stress conditions. Comparing with Eq. (62), one has

$$C_d = \frac{1}{1 - \frac{2}{9}\eta\eta_f}; \quad m = \eta_f \quad (68)$$

Figures 9 and 10 compare the experimental values of C_d at the maximum volume contraction during shearing with the predictions of the extended 3D stress-dilatancy formulation. Figure 9 shows that the extended 3D stress-dilatancy relation correctly reproduces the dependency of C_d , the reciprocal of the linear gradient of a dilatancy plot, on Lode's angle. At the same time, the model predictions for the shapes of the characteristic stress surfaces are in good agreement with experimental data.

By examining Eq. (67), when m and C_d are independent of stress ratio η , the initial rate of volume change at $\eta=0$ can be expressed as

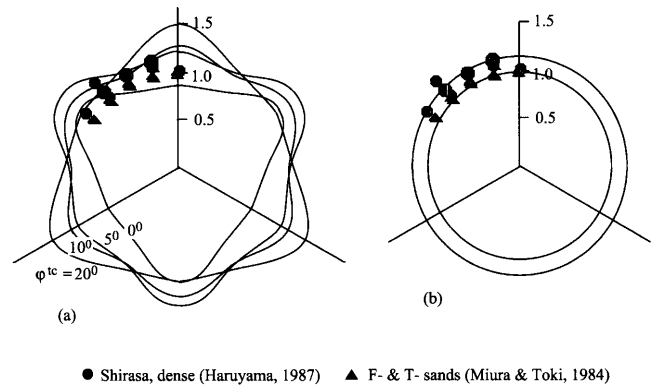


Fig. 11. The product of m and C_d on the π -plane: experimental results and model predictions

$$\frac{d\varepsilon_v}{d\gamma} = \frac{2}{3}mC_d \quad (69)$$

which reveals that the product of m and C_d represents the rate of volume change at very small stress ratios. Figure 11(a) compares mC_d from the data presented in Figs. 9 and 10 with model predictions using different stress ratio η , which is identified by the mobilized friction in triaxial compression conditions. One observes that the calculated products of m and C_d at very small stress ratio η are consistent with experimental data. It is worthwhile to mention that Miura and Toki proposed that the product of m and C_d in Eq. (67) should be constant, as shown in Fig. 11(b). However, given that mC_d as a constant, Eq. (67) yields the same initial rate of volume compaction when $\eta=0$ for any Lode's angle, and cannot describe the volume change characteristics shown in Figs. 5 through 7.

CLOSING REMARKS

Rowe's stress-dilatancy formulation has received considerable acceptance with respect to determining the volume change induced by shearing. This paper has extended Rowe's model to include more general 3D stress states. The underlying principle is that Rowe's hypothesis of minimum energy ratio can be expressed in terms of the ratio of spherical and deviatoric invariants of tensor π_{ij} . Comparisons with experimental data reveal that the extended 3D stress-dilatancy relation correctly captures the dilatancy behaviour of sand deforming along general 3D stress paths. Motivated by the success of this extension when dealing with coaxial case and principal stress/strain descriptions, the authors feel that the proposed method could be a reasonable extension to accommodate non-coaxial behaviour of granular materials.

ACKNOWLEDGEMENTS

The authors are grateful for funding received from the Natural Science and Engineering Research Council of Canada.

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