

## DETERMINATION OF CONSOLIDATION CONSTANTS OF SATURATED CLAYS

SEIKI OHMAKI\*

### ABSTRACT

A general theory for the three-dimensional deformation of porous materials is presented to formulate a boundary value problem. The theory deals with a two-phase material consisting of viscoelastic skeleton and incompressible water. The effective stress-strain relationship of the viscoelastic skeleton is assumed to follow the linear theory of viscoelasticity developed by Gurtin and Sternberg which is based on the Stieltjes convolution.

In the latter half of this paper, the triaxial compression creep test is analyzed as a boundary value problem. In the triaxial compression creep test a cylindrical sample is used which is subjected to the radial and axial compressive stresses under drained conditions. Drainage from a sample takes place only at its cylindrical wall and not on its ends. Under these conditions, the solution is obtained through the Laplace transformation. The deformation and the effective stress in a sample are obtained by applying the inverse Laplace transformation. The solution reveals that the deformation of a sample is composed of two parts, one is the immediate change which is caused by the deviator stress, the other is the time dependent change which is caused by the mean effective stress.

**Key words:** clay, consolidation, deformation, drained shear, laboratory test, test procedure, time effect, triaxial compression test

**IGC:** D5/E2

### INTRODUCTION

The three-dimensional deformation of a saturated clay is reduced to a problem of investigating the behaviour of a soil-water system consisting of two parts, the soil skeleton and the pore water. Ever since the one-dimensional consolidation theory based on the concept of effective stresses was established by Terzaghi (1924), a rapid advance has been made in this field of soil mechanics. Extension of the consolidation theory from one-dimensional analysis to three-dimensional analysis led to two analytical procedures. The one sets up a partial differential equation to solve for the excess pore water pressure alone (Rendulic, 1935; Schiffman, 1958) and the other solves simultaneous partial differential equations for the excess pore water pressure and the displacement of soil skeleton. (Biot, 1941).

For many problems of three-dimensional consolidation, the former treatment has been used by many investigators (Carrillo, 1942; Barron, 1948; Richart, 1957; Akai, 1953). In this case it is relatively easy to solve the problem because the only unknown is the excess pore water pressure, ignoring the deformation of soil skeleton. Meanwhile the analytical solution of the latter generally involves some difficulties, because partial differential equations have

\* Assistant, Disaster Prevention Research Institute, Kyoto University, Gokasho, Uji, Kyoto.

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to be solved for the excess pore water pressure and the deformation of soil skeleton. But this is a more general procedure to solve the consolidation problem because the behaviour of both the soil skeleton and the pore water can be obtained explicitly. This treatment becomes important in dealing with a two-phase porous material like a saturated clay in the analysis of settlement of or flow through soil layers.

McNamee and Gibson (1960 a, b) solved these equations for the plane strain and axisymmetric problems. And some solutions of boundary value problems were given by making use of Biot's theory (Gibson and McNamee, 1957; Schiffman and Fungaroli, 1965).

On the other hand, Taylor and Merchant (1940) pointed out that the secondary compression of a saturated clay was caused by the viscoelastic property of soil skeleton, considering that the time dependent deformation of a saturated clay is attributed to squeezing out the pore water and to the viscoelastic properties of soil skeleton. Biot (1954, 1956) also developed a consolidation theory of an anisotropic porous material, applying a linear theory of viscoelasticity. The linear theory of viscoelasticity is based on the assumption that a stimulus-response relationship of a material obeys the principle of superposition. Attempts have been made for the generalization of this theory by many investigators.

Therefore the three-dimensional consolidation theory pertaining to the secondary compression of a saturated clay should be obtained by combining Terzaghi's concept of effective stresses with the theory of viscoelasticity. But at the present time the viscoelastic theory is not fully developed to determine the consolidation constants of a saturated clay if not for the one-dimensional analysis.

This paper states firstly the three-dimensional consolidation theory of a two-phase porous material composed of the incompressible pore water and the viscoelastic soil skeleton, in which a linear theory of viscoelasticity developed by Gurtin and Sternberg (1962) is used to represent the effective stress-strain relationship. The theory is applied to solving an axisymmetric problem under the boundary conditions similar to those of a triaxial creep test of a saturated clay. Then the method of obtaining the consolidation constants of saturated clays from the triaxial drained creep test data is stated. The consolidation constants of Fujinomori clay are shown.

#### PROPOSED THREE-DIMENSIONAL CONSOLIDATION THEORY

##### *Assumptions Made for the Theory*

The soil-water system is assumed to have the following properties:

- 1) The soil-water system is isotropic and homogeneous. The strain, velocity and stress increments are sufficiently small and the theory is quasi-static.
- 2) The soil is completely saturated with the incompressible water.
- 3) The flow of pore water through the soil-water system obeys Darcy's law and the coefficient of permeability is isotropic.
- 4) The porosity of soil-water system is defined as

$$\lim_{\Delta V \rightarrow 0} \frac{\Delta V_v}{\Delta V}$$

where  $V_v$  and  $V$  are the pore and total volumes under consideration.

- 5) The relationship between the pore water pressure and the stresses is given in rectangular Cartesian coordinates as

$$\tau_{ij} = \tau'_{ij} + u \delta_{ij} \quad (1)$$

where  $\tau_{ij}$  and  $\tau'_{ij}$  are the total and the effective stresses,  $u$  is the pore water pressure and

$\delta_{ij}$  is the Kronecker's delta. The positive direction of each stress on the  $x_1$ -plane is shown in Fig. 1. It is determined on other planes in a similar manner. (Compression is assumed to be positive.)

6) The effective stress-strain relationship of soil skeleton is linearly viscoelastic; namely in a homogeneous material it is represented as follows;

The creep integral law,

$$\left. \begin{aligned} \varepsilon_{ij} &= s_{ij} * dJ_1 \\ e_{kk} &= \tau_{kk}' * dJ_2 \end{aligned} \right\} \quad (2)$$

The relaxation integral law,

$$\left. \begin{aligned} s_{ij} &= \varepsilon_{ij} * dG_1 \\ \tau_{kk}' &= e_{kk} * dG_2 \end{aligned} \right\} \quad (3)$$

The differential operator law,

$$\left. \begin{aligned} P_1(D) s_{ij} &= Q_1(D) \varepsilon_{ij} \\ P_2(D) \tau_{kk}' &= Q_2(D) e_{kk} \end{aligned} \right\} \quad (4)$$

where  $e_{ij}$  is a strain tensor, the direction of which depends on the corresponding stress tensor.

The positive sign indicates the compression.

$$e_{ij} = -\frac{1}{2}(u_{i,j} + u_{j,i})$$

where  $u_i$  are the components of the displacement of soil skeleton. In Eqs. (2), (3) and (4),  $\varepsilon_{ij}$  and  $s_{ij}$  are the deviator components of the strain and the stress tensors, namely

$$\left. \begin{aligned} \varepsilon_{ij} &= e_{ij} - \frac{1}{3} \delta_{ij} e_{kk} \\ s_{ij} &= \tau_{ij}' - \frac{1}{3} \delta_{ij} \tau_{kk}' \end{aligned} \right\} \quad (5)$$

$J_1$  and  $J_2$  are the creep functions and  $G_1$ ,  $G_2$  are the relaxation functions in shear and in isotropic compression, respectively.

The asterisk represents the Stieltjes convolution, that is,

$$\varphi * d\psi = \int_{-\infty}^t \varphi(t-\tau) d\psi(\tau)$$

Especially when  $\psi = \varphi = 0$  for  $t < 0$ ,

$$\begin{aligned} \varphi * d\psi &= \psi(0+) \varphi(t) + \int_0^t \varphi(t-\tau) \frac{d\psi(\tau)}{d\tau} d\tau \\ & (= \psi(0+) \varphi(t) + \varphi * \psi^{(1)}) \end{aligned}$$

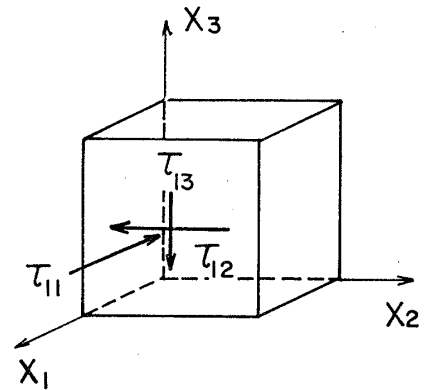


Fig. 1. Directions of stresses

Now, notation  $D$  represents  $\frac{\partial}{\partial t}$  and  $P_i(D)$ ,  $Q_i(D)$  are polynomials of  $D$ . It is also possible to specify the relationships between  $P_i(D)$ ,  $Q_i(D)$  and  $J_i$  or  $G_i$ . ( $i=1, 2$ ) Therefore it is easy to understand that effective stress-strain relationship represented by linear viscoelastic models such as a Maxwell or Voigt model or their arbitrary combinations can be expressed by Stieltjes convolution.

### Equations of Equilibrium

Equations of equilibrium are given

$$\tau_{ij, j} = F_i \quad (6)$$

where  $F_i$  is a component of the body force. The body force of a clay with porosity  $n$ , is given as

$$F_i = -\{(1-n)\gamma_s + n\gamma_w\} \mathbf{n} \cdot \mathbf{e}_i$$

where  $\mathbf{n}$ ,  $\mathbf{e}_i$  are the unit vectors in the vertically upward and in the  $i$ -coordinate axis directions and  $\gamma_s$ ,  $\gamma_w$  are the weights per unit volume of the soil grains and of the water, meanwhile the porosity  $n$  is defined by

$$n = \frac{n_0 - e_v}{1 - e_v}$$

where  $n_0$  represents the initial porosity, therefore

$$F_i = -\frac{1}{1 - e_v} \{(1 - n_0)\gamma_s + (n_0 - e_v)\gamma_w\} \mathbf{n} \cdot \mathbf{e}_i \quad (7)$$

### Equation of Continuity

When  $h_e$  and  $H$  are defined as indicated in Fig. 2, then

$$u = \gamma_w(H - \mathbf{n} \cdot \mathbf{r}) + u_e$$

where  $\mathbf{r}$  and  $u_e$  represent the position vector and the excess pore water pressure. Denoting the weight per unit volume of the soil-water system as  $\gamma$ ,

$$\gamma = (1 - n)\gamma_s + n\gamma_w \quad (8)$$

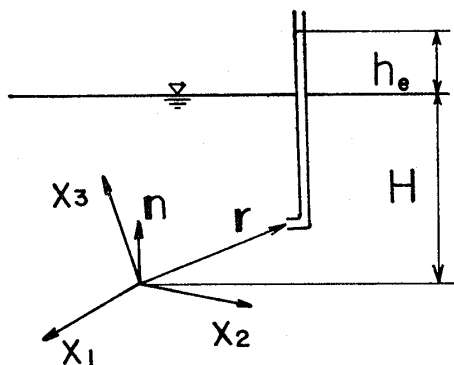


Fig. 2. A piezometric tube in the soil

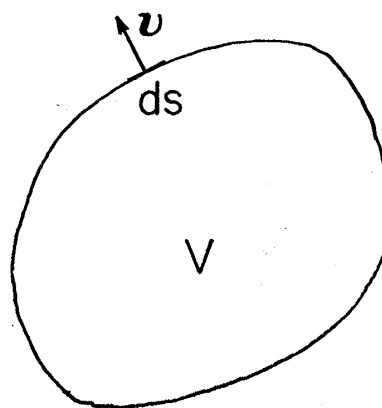


Fig. 3. Arbitrary closed volume within soil mass

Now the equation of continuity is considered over the volume  $V$  and on its surface  $S$  as indicated in Fig. 3. Letting the displacement vectors of the soil skeleton and the pore water be  $\mathbf{u}$  and  $\mathbf{U}$  and the corresponding rate vectors be  $\dot{\mathbf{u}}$  and  $\dot{\mathbf{U}}$ , the weight of soil and water flowing out of  $V$  is

$$\int_S \{(1-n)\gamma_s \dot{\mathbf{u}} + n\gamma_w \dot{\mathbf{U}}\} \cdot \mathbf{v} dS$$

where  $\mathbf{v}$  represents a unit outward normal vector of  $V$ .

This must be equal to the decrease in weight from volume  $V$  per unit time

$$-\int_V \dot{\gamma} dV$$

Applying the Green-Gauss theorem for converting the surface integral into the volume integral to vanish the integrand,

$$\nabla \cdot \{(1-n)\gamma_s \dot{\mathbf{u}} + n\gamma_w \dot{\mathbf{U}}\} + \dot{\gamma} = 0$$

Using Eq. (8) and Darcy's law;

$$n\dot{\mathbf{U}} - \dot{\mathbf{u}} = -\frac{k}{\gamma_w} \nabla u_e$$

where  $k$  is the coefficient of permeability, then

$$\nabla \cdot \left\{ \gamma_s (1-n) \dot{\mathbf{u}} + \gamma_w \left( \dot{\mathbf{u}} - \frac{k}{\gamma_w} \nabla u_e \right) \right\} - (\gamma_s - \gamma_w) \dot{n} = 0$$

Considering the relationship

$$n = \frac{n_0 - e_v}{1 - e_v} \simeq n_0 - e_v(1 - n_0), \quad \dot{n} = -(1 - n_0)\dot{e}_v, \quad \nabla k = 0, \quad \nabla n_0 = 0,$$

then

$$\nabla \dot{\mathbf{u}} - \frac{k}{\alpha} \nabla^2 u_e = 0 \quad (9)$$

where

$$\alpha = (2 - n_0)\gamma_w$$

This is the equation of continuity, which is similar to that obtained hitherto by Biot and others, with a slightly different form of the coefficient of consolidation.

#### *Basic Equation of Consolidation Concerning Displacement Vector and Excess Pore Water Pressure*

From Eq. (1),

$$\tau_{ij, j} = \tau'_{ij, j} + u_{, i}$$

Using relationships of Eqs. (3) and (5),

$$\tau_{ij}' = e_{ij} * dG_1 + \frac{1}{3} \delta_{ij} e_{kk} * d(G_2 - G_1)$$

Now substituting this equation and Eq. (6) into the former,

$$e_{ij, j} * dG_1 + \frac{1}{3} e_{kk, i} * d(G_2 - G_1) + u_{,i} = F_i$$

This equation is expressed in terms of the displacement vector of soil-skeleton and the excess pore water pressure,

$$\nabla^2 \mathbf{u} * dG_1 + \nabla \nabla \cdot \mathbf{u} * dK - 2\nabla u_e + 2\gamma_w \nabla (\mathbf{n} \cdot \mathbf{r}) + 2\mathbf{F} = 0 \quad (10)$$

where  $H$  is assumed constant and  $K = (G_1 + 2G_2)/3$ . Solutions of boundary value problems of the three-dimensional consolidation can be obtained by solving Eqs. (9) and (10) simultaneously under the given boundary conditions.

#### BEHAVIOURS OF TWO-PHASE MATERIAL WITH AXISYMMETRIC BOUNDARY CONDITION

In the former section, basic Eqs. (9) and (10) governing the three-dimensional consolidation have been obtained. In this section, their solutions are given for the two-phase cylindrical material which is subjected to a constant load for a long duration, as in the case of a clay sample in the triaxial creep test.

#### Laplace Transformation of Basic Equations for the Axisymmetric Condition and their Solutions

Using nondimensional variables,

$$\bar{u}_1 = \frac{u_1}{a}, \quad \bar{u}_3 = \frac{u_3}{h}, \quad \bar{u}_e = \frac{u_e}{\alpha h}, \quad r = \frac{x_1}{a} \quad \text{and} \quad z = \frac{x_3}{h}$$

where  $x_1$  and  $x_3$  are the cylindrical coordinates indicated in Fig. 4,  $u_1$  and  $u_3$  are the real  $x_1$ - and  $x_3$ -components of the displacement of soil skeleton, and  $a$  and  $h$  are the radius and the height of a cylindrical sample, then Eq. (10) is written as

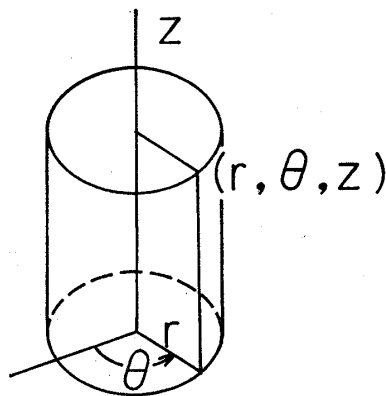


Fig. 4. Cylindrical coordinate

$$\left. \begin{aligned} \left( \frac{\partial^2 \bar{u}_1}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{u}_1}{\partial r} + \frac{a^2}{h^2} \frac{\partial^2 \bar{u}_1}{\partial z^2} - \frac{\bar{u}_1}{r^2} \right) * dG_1 - \frac{\partial \bar{e}_v}{\partial r} * dK - 2\alpha h \frac{\partial \bar{u}_e}{\partial r} &= 0 \\ \left( \frac{h^2}{a^2} \frac{\partial^2 \bar{u}_3}{\partial r^2} + \frac{h^2}{a^2} \frac{\partial \bar{u}_3}{r \partial r} + \frac{\partial^2 \bar{u}_3}{\partial z^2} \right) * dG_1 - \frac{\partial \bar{e}_v}{\partial z} * dK - 2\alpha h \frac{\partial \bar{u}_e}{\partial z} &= 0 \end{aligned} \right\} \quad (11)$$

where the term  $2\gamma_w \nabla(\mathbf{n} \cdot \mathbf{r}) + 2\mathbf{F}$  is neglected. Eliminating the excess pore water pressure  $u_e$  from Eqs. (9) and (10),

$$\left. \begin{aligned} & \left( \frac{\partial^2 \bar{e}_v}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{e}_v}{\partial r} + \frac{a^2}{h^2} \frac{\partial^2 \bar{e}_v}{\partial z^2} \right) * dL - \frac{a^2 \alpha}{k} \frac{\partial \bar{e}_v}{\partial t} = 0 \\ & \text{and} \\ & \frac{\partial^2 \bar{u}_e}{\partial r^2} + \frac{1}{r} \frac{\partial \bar{u}_e}{\partial r} + \frac{a^2}{h^2} \frac{\partial^2 \bar{u}_e}{\partial z^2} = - \frac{a^2}{kh} \frac{\partial \bar{e}_v}{\partial t} \end{aligned} \right\} \quad (12)$$

where

$$\left. \begin{aligned} \bar{e}_v &= - \left( \frac{\partial \bar{u}_1}{\partial r} + \frac{\bar{u}_1}{r} + \frac{\partial \bar{u}_3}{\partial z} \right) \\ \bar{\tau}_{11}' &= - \frac{\partial \bar{u}_1}{\partial r} * dG_1 + \frac{1}{3} \bar{e}_v * d(G_2 - G_1) \\ \bar{\tau}_{33}' &= - \frac{\partial \bar{u}_3}{\partial z} * dG_1 + \frac{1}{3} \bar{e}_v * d(G_2 - G_1) \\ \bar{\tau}_{13}' &= - \left( \frac{a}{h} \frac{\partial \bar{u}_1}{\partial z} + \frac{h}{a} \frac{\partial \bar{u}_3}{\partial r} \right) * dG_1 \end{aligned} \right\} \quad (13)$$

and

$$L = \frac{1}{3} (2G_1 + G_2)$$

The boundary conditions are

$$\left. \begin{aligned} \bar{\tau}_{11}' = p, \bar{\tau}_{13}' = 0, \bar{u}_e = 0, \quad \text{for } r=1, 0 \leq z \leq 1, 0 \leq t < \infty \\ \bar{u}_3 = 0, \bar{\tau}_{13}' = 0, \frac{\partial \bar{u}_e}{\partial z} = 0, \quad \text{for } 0 \leq r \leq 1, z=0, 0 \leq t < \infty \\ \frac{\partial \bar{u}_e}{\partial z} = 0, \bar{\tau}_{13}' = 0, 2\pi \int_0^1 (\bar{\tau}_{33}' + \alpha h \bar{u}_e) r dr = \pi q, \quad \text{for } 0 \leq r \leq 1, z=1, 0 \leq t < \infty \end{aligned} \right\} \quad (14)$$

To simplify the treatment of Stieltjes convolution in Eqs. (11), (12) and (13), the Laplace transformations are used hereafter.

The Laplace transformation of the Stieltjes convolution of two functions  $\varphi(t)$  and  $\psi(t)$  is given as

$$L(\varphi(t) * d\psi(t)) = \overline{\varphi(s)} \cdot s \overline{\psi(s)}$$

where  $\overline{\varphi(s)}$  and  $\overline{\psi(s)}$  are the Laplace transformations of  $\varphi(t)$  and  $\psi(t)$ , namely

$$\overline{\varphi(s)} = L\varphi(t) = \int_0^\infty \varphi(t) e^{-st} dt$$

then Eqs. (11) and (12) are given as follows:

$$\left. \begin{aligned}
 \frac{\partial^2 V_1}{\partial r^2} + \frac{1}{r} \frac{\partial V_1}{\partial r} - \frac{V_1}{r^2} + \frac{a^2}{h^2} \frac{\partial^2 V_1}{\partial z^2} &= \frac{\partial}{\partial r} (PE + Q\Phi) \\
 \frac{\partial^2 V_3}{\partial r^2} + \frac{1}{r} \frac{\partial V_3}{\partial r} + \frac{a^2}{h^2} \frac{\partial^2 V_3}{\partial z^2} &= \frac{a^2}{h^2} \frac{\partial}{\partial z} (PE + Q\Phi) \\
 \frac{\partial^2 E}{\partial r^2} + \frac{1}{r} \frac{\partial E}{\partial r} + \frac{a^2}{h^2} \frac{\partial^2 E}{\partial z^2} &= \lambda^2 E \\
 \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{a^2}{h^2} \frac{\partial^2 \Phi}{\partial z^2} &= -\frac{a^2 s}{kh} E
 \end{aligned} \right\} \quad (15)$$

where  $V_1$ ,  $V_3$ ,  $E$  and  $\Phi$  are the Laplace transformations of  $u_1$ ,  $u_3$ ,  $e_r$  and  $u_e$  respectively, and  $\bar{G}_1$  and  $\bar{G}_2$  are also Laplace transformations of  $G_1(t)$  and  $G_2(t)$  and where,

$$P = \frac{\bar{G}_1 + 2\bar{G}_2}{3\bar{G}_1}, \quad Q = \frac{2\alpha h}{s\bar{G}_1}, \quad \lambda^2 = \frac{3a^2\alpha}{k(2\bar{G}_1 + \bar{G}_2)}$$

On the other hand, the boundary conditions are,

for  $r=1, 0 \leq z \leq 1$

$$T_{11}' = \frac{p}{s}, \quad T_{13}' = 0, \quad \Phi = 0$$

for  $0 \leq r \leq 1, z=0$

$$V_3 = 0, \quad T_{13}' = 0, \quad \frac{\partial \Phi}{\partial z} = 0$$

for  $0 \leq r \leq 1, z=1$

$$2\pi \int_0^1 (T_{33}' + \alpha h \Phi) r dr = \frac{\pi q}{s}, \quad T_{13}' = 0, \quad \frac{\partial \Phi}{\partial z} = 0$$

and

$$\left. \begin{aligned}
 T_{11}' &= -s\bar{G}_1 \frac{\partial V_1}{\partial r} + \frac{s}{3} (\bar{G}_2 - \bar{G}_1) E \\
 T_{33}' &= -s\bar{G}_1 \frac{\partial V_3}{\partial z} + \frac{s}{3} (\bar{G}_2 - \bar{G}_1) E \\
 T_{13}' &= -s\bar{G}_1 \left( \frac{a}{h} \frac{\partial V_1}{\partial z} + \frac{h}{a} \frac{\partial V_3}{\partial r} \right)
 \end{aligned} \right\} \quad (17)$$

A solution of Eq. (15) under the boundary conditions Eq. 16 is given as follows (Ohmaki, 1971);

$$\left. \begin{aligned}
 V_1 &= \frac{q-p}{3s^2\bar{G}_1} r + \frac{2p+q}{3} \frac{I_1(\lambda)r - 3I_1(\lambda r)}{s^2\Lambda} \\
 V_3 &= -\frac{2(q-p)}{3s^2\bar{G}_1} z - \frac{2p+q}{3} \frac{2I_1(\lambda)}{s^2\Lambda} z \\
 E &= (2p+q) \frac{\lambda I_0(\lambda r)}{s^2\Lambda} \\
 \Phi &= \frac{2p+q}{3} \frac{(2\bar{G}_1 + \bar{G}_2)\lambda \{I_0(\lambda) - I_0(\lambda r)\}}{\alpha h s \Lambda}
 \end{aligned} \right\} \quad (18)$$

and

$$\left. \begin{aligned} T_{11}' &= \frac{p-q}{3s} + \frac{2p+q}{3} \frac{(2\bar{G}_1 + \bar{G}_2)\lambda I_0(\lambda r) - \bar{G}_1 \left\{ I_1(\lambda) + \frac{3I_1(\lambda r)}{r} \right\}}{sA} \\ T_{33}' &= \frac{2(q-p)}{3s} + \frac{2p+q}{3} \frac{\{(\bar{G}_2 - \bar{G}_1)\lambda I_0(\lambda r) + 2\bar{G}_1 I_1(\lambda)\}}{sA} \\ T_{13}' &= 0 \end{aligned} \right\} \quad (19)$$

In these equations,  $V_1$ ,  $V_3$ ,  $T_{11}'$  and  $T_{33}'$  are divisible into two components, that is, the one which is proportional to the deviator stress  $(q-p)$  (in future, this will be called the deviator component) and the other which is proportional to the mean stress  $(2p+q)/3$  (this will be called the isotropic component).  $V_1$ ,  $E$ ,  $\Phi$ ,  $T_{11}'$  and  $T_{33}'$  are the functions of  $r$  only and independent of  $z$ , and  $V_3$  is proportional to  $z$ .

#### Solutions for Constants $G_1$ and $G_2$

In a simple case, where  $G_1$  and  $G_2$  are considered constant with time, solutions are obtained by applying the inverse theorem of Laplace transformation to Eq. (18).

Assuming  $G_1$  and  $G_2$  are constant in  $[0, \infty)$ , these Laplace transformations are

$$\bar{G}_1 = \frac{G_1}{s}, \quad \bar{G}_2 = \frac{G_2}{s} \quad (Rs > 0)$$

then Eqs. (18) and (19) are rewritten as

$$\left. \begin{aligned} V_1 &= \frac{q-p}{3sG_1} r + \frac{2p+q}{3} \frac{I_1(\lambda)r - 3I_1(\lambda r)}{s\Pi} \\ V_3 &= -\frac{2(q-p)}{3sG_1} z - \frac{2p+q}{3} \frac{2I_1(\lambda)}{s\Pi} z \\ E &= (2p+q) \frac{\lambda I_0(\lambda r)}{s\Pi} \\ \Phi &= \frac{2p+q}{3} \frac{\lambda(2G_1 + G_2) \{I_0(\lambda) - I_0(\lambda r)\}}{\alpha h s \Pi} \\ T_{11}' &= \frac{p-q}{3s} + \frac{2p+q}{3} \frac{(2G_1 + G_2)\lambda I_0(\lambda r) - G_1 \left\{ I_1(\lambda) + \frac{3I_1(\lambda r)}{r} \right\}}{s\Pi} \\ T_{33}' &= \frac{2(q-p)}{3s} + \frac{2p+q}{3} \frac{\{(G_2 - G_1)\lambda I_0(\lambda r) + 2G_1 I_1(\lambda)\}}{s\Pi} \end{aligned} \right\} \quad (20)$$

where

$$\Pi = (2G_1 + G_2)\lambda I_0(\lambda) - 4G_1 I_1(\lambda)$$

$$\lambda^2 = \frac{3a^2\alpha}{k(2G_1 + G_2)} s$$

The inverse transformations of the displacements and the stresses in Eq. (20) are given as follows (Ohmaki, 1971);

$$\begin{aligned}
 \bar{u}_1 &= \frac{q-p}{3G_1} r - \frac{2p+q}{3G_2} r - \frac{2p+q}{3} \sum_{n=1}^{\infty} \frac{8G_1}{\nu_n \Theta_n} \frac{J_1(\nu_n)r - 3J_1(\nu_n r)}{J_0(\nu_n)} \exp(-\nu_n^2 T) \\
 \bar{u}_3 &= -\frac{2(q-p)}{3G_1} z - \frac{2p+q}{3G_2} z + \frac{2p+q}{3} \sum_{n=1}^{\infty} \frac{16G_1}{\nu_n \Theta_n} \frac{J_1(\nu_n)}{J_0(\nu_n)} \exp(-\nu_n^2 T) z \\
 \bar{e}_v &= \frac{2p+q}{G_2} - (2p+q) \sum_{n=1}^{\infty} \frac{8G_1}{\Theta_n} \frac{J_0(\nu_n r)}{J_0(\nu_n)} \exp(-\nu_n^2 T) \\
 \bar{u}_e &= -\frac{2p+q}{3\alpha h} \sum_{n=1}^{\infty} \frac{8G_1(2G_1+G_2)}{\Theta_n} \frac{J_0(\nu_n) - J_0(\nu_n r)}{J_0(\nu_n)} \exp(-\nu_n^2 T) \\
 \bar{\tau}_{11}' &= p + \frac{2p+q}{3} \sum_{n=1}^{\infty} \frac{8G_1}{\Theta_n} \frac{G_1 \frac{J_1(\nu_n)}{\nu_n} + 3G_1 \frac{J_1(\nu_n r)}{\nu_n r} - (2G_1+G_2) J_0(\nu_n r)}{J_0(\nu_n)} \exp(-\nu_n^2 T) \\
 \bar{\tau}_{33}' &= q + \frac{2p+q}{3} \sum_{n=1}^{\infty} \frac{8G_1}{\Theta_n} \frac{-2G_1 \frac{J_1(\nu_n)}{\nu_n} + (G_1-G_2) J_0(\nu_n r)}{J_0(\nu_n)} \exp(-\nu_n^2 T)
 \end{aligned} \tag{21}$$

where

$$\Theta_n = (2G_1 + G_2)^2 \nu_n^2 - 8G_1 G_2$$

$$T = \frac{k(2G_1 + G_2)}{3a^2 \alpha} \Big|_t$$

#### *Characteristics of Solutions Obtained and Method of Determining the Consolidation Constants*

Now the characteristics of the solutions obtained will be discussed.

Using the relationship,

$$(2G_1 + G_2) \nu J_0(\nu) - 4G_1 J_1(\nu) = 0 \tag{22}$$

and Dini's expression by Bessel functions, then

$$\frac{2p+q}{G_2} r = (2p+q) \sum_{n=1}^{\infty} \frac{16G_1 J_1(\nu_n r)}{\nu_n \Theta_n J_0(\nu_n)}$$

Differentiating both terms with respect to  $r$

$$\frac{2p+q}{G_2} = (2p+q) \sum_{n=1}^{\infty} \frac{8G_1 J_0(\nu_n r)}{\Theta_n J_0(\nu_n)} \tag{23}$$

From this relation  $e_v=0$  in the third expression of Eq. (21) when  $t=0$ , namely there is no volume change all over the sample immediately after loading.

$$\begin{aligned}
 & \frac{2p+q}{3} \sum_{n=1}^{\infty} \frac{8G_1}{\nu_n \Theta_n} \frac{J_1(\nu_n)r - J_1(\nu_n r)}{J_0(\nu_n)} \\
 &= \frac{2p+q}{3} r \sum_{n=1}^{\infty} \frac{8G_1}{\nu_n \Theta_n} \frac{J_1(\nu_n x)}{J_0(\nu_n)} \Big|_{x=1} - (2p+q) \sum_{n=1}^{\infty} \frac{8G_1}{\nu_n \Theta_n} \frac{J_1(\nu_n r)}{J_0(\nu_n)} \\
 &= -\frac{2p+q}{3G_2} r
 \end{aligned}$$

then from the first equation in Eq. (21),

$$\left. \begin{aligned} \bar{u}_1 &= \frac{q-p}{3G_1} r, & \text{when } T=0 \\ \bar{u}_3 &= -\frac{2(q-p)}{3G_1} r, & \text{when } T=0 \end{aligned} \right\} \quad (24)$$

Similarly,

Therefore, the instantaneous deformation of the sample or  $\bar{u}_1$  and  $\bar{u}_3$  at  $T=0$  are given by the deviator stress  $q-p$  only. That means, these deformations results from the shear stress alone. Since there is no volumetric change at  $T=0$ , these deformations should coincide with those obtained from the linear theory of elasticity under the same boundary conditions, in which the apparent Young's modulus  $E_u = \frac{3}{2} G_1$  and the Poisson's ratio  $\mu_u = \frac{1}{2}$  as indicated by Mikasa (1951). In the process of consolidation, the isotropic components of stresses, the deformations and the excess pore water pressure varies with time.

Eventually when  $T$  goes to infinity,

$$\left. \begin{aligned} \bar{u}_1 &= \frac{q-p}{3G_1} r - \frac{2p+q}{3G_2} r \\ \bar{u}_3 &= -\frac{2(q-p)}{3G_1} z - \frac{2p+q}{3G_2} z \\ \bar{e}_v &= \frac{2p+q}{G_2}, \quad \bar{u}_e = 0 \\ \bar{\tau}_{11}' &= p, \quad \bar{\tau}_{33}' = q, \quad \bar{\tau}_{13}' = 0 \end{aligned} \right\} \quad (25)$$

In these equations, both  $\bar{u}_1$  and  $\bar{u}_3$  are composed of the isotropic and deviator stress components and proportional to  $r$  and  $z$  respectively. If  $q > p$ , the soil sample undergoes an instantaneous expansion in the radial direction and a contraction in the axial direction. In the course of consolidation, the sample contracts in both directions.  $\bar{\tau}_{11}'$  and  $\bar{\tau}_{33}'$  converge to the constants  $p$  and  $q$  respectively.

#### METHOD OF DETERMINING THE CONSOLIDATION CONSTANTS AND TEST RESULTS

The consolidation constants are determined easily from the characteristics of the solutions derived above. Usually in the triaxial drained creep test it is possible to measure the volume of expelled water from the specimen and the settlement of the upper side of the specimen. Therefore substituting the settlement of the upper side of the sample immediately after loading into the second equation of Eq. (24),  $G_1$  is obtained,

$$G_1 = \frac{2(q-p)}{3} \frac{h}{u_{3i}}$$

where  $u_{3i}$  is the settlement of the upper side of the specimen. And integrating the third equation in Eq. (21) over the sample,

$$\frac{\Delta V}{V} = (2p+q) \left\{ \frac{1}{G_2} - \sum_{n=1}^{\infty} \frac{16G_1}{\nu_n \theta_n} \frac{J_1(\nu_n)}{J_0(\nu_n)} e^{-\nu_n^2 T} \right\} \quad (26)$$

where  $\Delta V$  and  $V$  are the volume of the expelled water and the initial volume of sample. If  $T$  is sufficiently large,

$$\frac{\Delta V}{V} \rightarrow \frac{2p+q}{G_2} = \left( \frac{\Delta V}{V} \right)_f$$

then

$$G_2 = \frac{2p+q}{\left( \frac{\Delta V}{V} \right)_f}$$

therefore this equation gives  $G_2$  in terms of the volume of expelled water. Calculating the zero points  $\nu_n$  of  $\Pi$  from Eq. (22) to put them into Eq. (26), the coefficient of permeability  $k$  can be obtained from  $\frac{\Delta V}{V} \sim \log t$  plot in the same manner as the Casagrande's method for the one-dimensional consolidation.

In the following, the method of determining the consolidation constants is applied to the triaxial drained creep tests on a clay. The clay used in the tests came from Fujinomori Kyoto Prefecture. Its properties are given in Table 1. The cylindrical samples were 3.5 cm in radius, 7.9 cm high. Samples were consolidated isotropically at consolidation pressures of 0.5 kg/cm<sup>2</sup> and 1.0 kg/cm<sup>2</sup> for two days in each step, and thereafter loaded at the stresses as indicated in Table 2 and Fig. 5, in which axial and radial stresses were such

Table 1. Properties of clay

$\gamma_s$	2.68 g/cm <sup>3</sup>	>50 $\mu$	21.0%
L.L.	43.6%	50-5 $\mu$	40.0%
P.I.	17.5%	5 $\mu$ >	39.0%

Table 2. Conditions and results of tests

Test No.	$\sigma_1$ (kg/cm <sup>2</sup> )	$\sigma_3$ (kg/cm <sup>2</sup> )	$\sigma_m$ (kg/cm <sup>2</sup> )	$G_1$ (kg/cm <sup>2</sup> )	$G_2$ (kg/cm <sup>2</sup> )	$k$ (cm/sec)
No. 1	2.000	2.000	2.000	—	123.7	$1.16 \times 10^{-6}$
No. 2	2.250	1.875	2.000	89.7	144.6	3.38
No. 3	2.500	1.750	2.000	64.9	120.7	2.54
No. 4	2.750	1.625	2.000	Failure	—	—
No. 5	3.000	3.000	3.000	—	126.1	2.90
No. 6	3.375	2.8125	3.000	49.1	120.3	2.73
No. 7	3.750	2.625	3.000	Failure	—	—
No. 8	4.000	4.000	4.000	—	132.6	2.79
No. 9	4.500	3.750	4.000	50.5	118.5	1.52
No. 10	4.750	3.625	4.000	Failure	—	—

that the mean stress  $\sigma_m = (\sigma_1 + 2\sigma_3)/3$  was kept constant in every set of tests. Values of  $G_1$ ,  $G_2$  and  $k$  obtained by this method are also shown in Table 2. In Fig. 6 the experimental volumetric strain  $\frac{\Delta V}{V}$  versus the logarithm of time relationships are compared with the theoretical curves. Using the consolidation constants given in Table 2 the theoretical curves

were calculated. In this figure, a considerable discrepancy between the experimental plots and the theoretical curves is observed for the time interval from 100 minutes to 3000 minutes after loading. This indicates that the relaxation function for the isotropic compression of this clay should include a Voigt model in series in addition to a spring. In Fig. 7, comparison of plots of the experimental vertical strains  $\bar{u}_3$  versus the logarithm of time is made with the theoretical curves. In the calculation of  $\bar{u}_3$ , the mean value of  $G_1$  obtained from the tests under the deviator stress condition is assumed for the isotropic stress condition, because  $G_1$  is determinable only under the deviator stress condition. From this figure it is observed that the experimental vertical strain under the deviator stress condition is larger than the theoretical strain, and this indicates also that the relaxation function in shear cannot be expressed only by a constant determined from the initial settlement of the top of the specimen.

Therefore a viscoelastic model representing the time dependent behaviour of the specimen in shear is necessary.

From Table 2 it will be seen in the case of isotropic loading that the values of  $G_2$  increase slightly with the increase in the mean principal stress  $\sigma_m$ . Moreover the values of  $G_2$  obtained for the anisotropic loading are smaller than those for the isotropic loading,

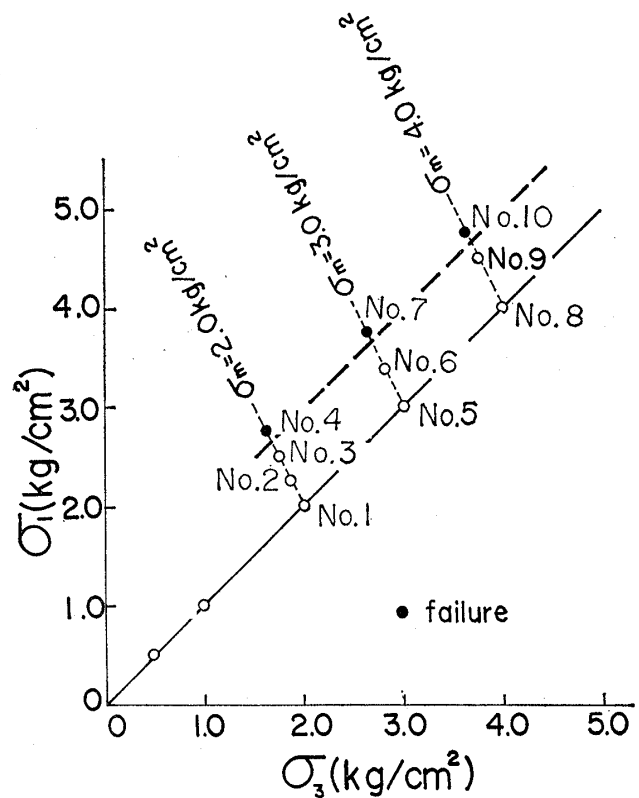


Fig. 5. Stress conditions in tests

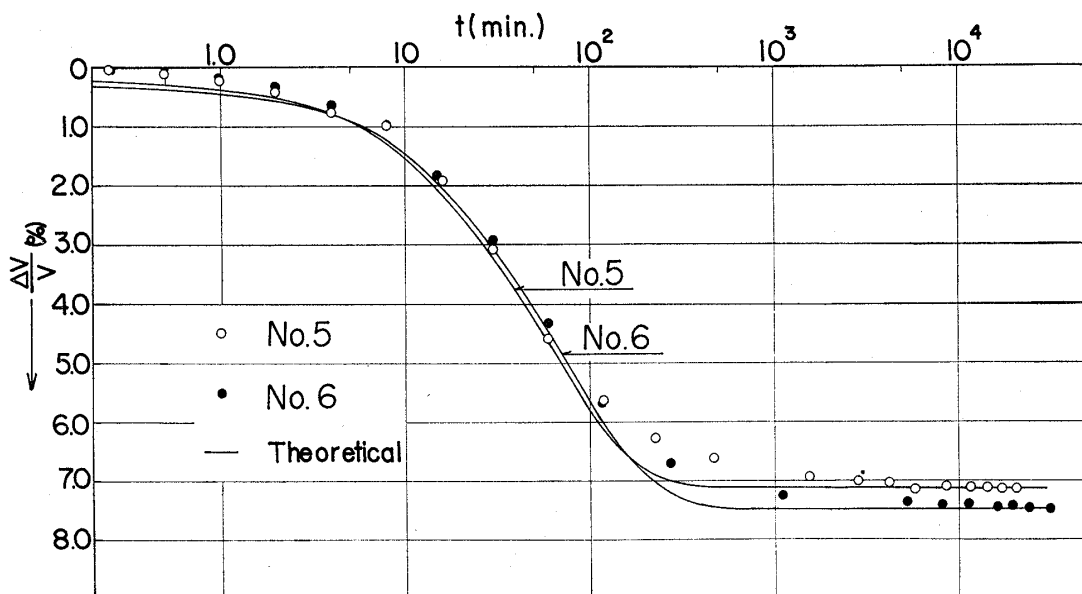


Fig. 6. Volumetric strain  $\sim \log t$  curves ( $\sigma_m = 3.0 \text{ kg/cm}^2$ )

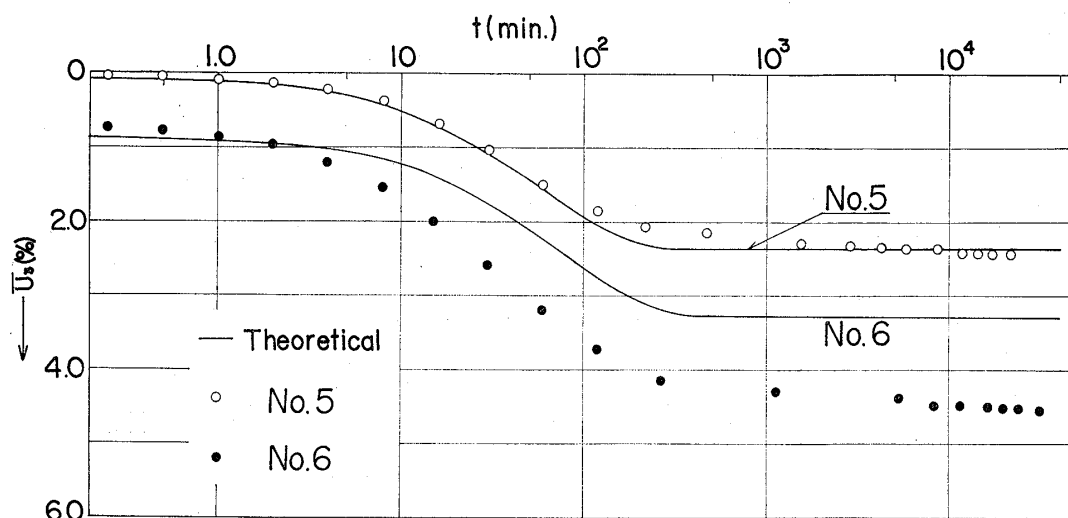


Fig. 7. Vertical strain  $\sim \log t$  curves ( $\sigma_m = 3.0 \text{ kg/cm}^2$ )

except for the test No. 2, when comparisons are made for the same stress level of  $\sigma_m$ . This fact indicates the effect of dilatancy of clay. On the other hand, although there is considerable scatter in the values of  $G_1$  indicated in Table 2, it may be attributed to the difficulty of measuring the displacement of the upper side of the specimen immediately after loading, and it is concluded that the value of  $G_1$  of this clay consolidated at the isotropic pressure of  $1.0 \text{ kg/cm}^2$  is within the range from  $50 \text{ kg/cm}^2$  to  $100 \text{ kg/cm}^2$ . Finally it is of interest to note that the failure line for these creep tests, which is shown by a dashed line in Fig. 5, is parallel to the isotropic stress line.

#### CONCLUSIONS

A three-dimensional consolidation theory for saturated clays has been obtained by using the Stieltjes convolution to represent the effective stress-strain relationship. And a boundary value problem has been solved under the boundary conditions of the triaxial drained creep test to determine the consolidation constants of saturated clays. Then, triaxial drained creep tests of Fujinomori clay have been carried out under the condition that the mean principal stress was constant in every set of tests.

From the analysis of the theory and the experimental results, it is concluded that:

1) Basic equations governing the consolidation are expressed in such a way that their forms resemble the case in which the soil skeleton has the elastic property, therefore the Laplace transformations can be applied likewise in solving these equations for stresses and strains in the sample.

2) Both of the calculated relaxation functions  $G_1(t)$  and  $G_2(t)$  for the shear and for the isotropic compression are independent of time and account for the following: a) radial and axial displacements are composed of two components, one proportional to the deviator stress which is produced immediately after loading and independent of time and the other proportional to the mean stress which is dependent of time and zero at the instance of loading; b) volume change is composed only of the term proportional to the mean stress; and c) finally the displacements converge to values consisting of two terms proportional to the deviator and the mean stresses, the volume change converges to only one term proportional to the mean stress.

3) Comparison of the experimental data and the theoretical calculation indicates that

time dependent models are necessary in both the isotropic compression and the shear to represent the behaviour of soil skeleton.

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## NOTATION

- $a$  = radius of cylindrical soil specimen  
 $e_v, \bar{e}_v$  = volumetric strain ( $e_v = \bar{e}_v$ )  
 $e_{ij} = ij$  component of strain tensor of soil skeleton  
 $E$  = Laplace transformation of  $\bar{e}_v$   
 $F_i = i$  component of body force vector  
 $G_1, G_2$  = relaxation function or elastic constants in shear and in isotropic compression  
 $\bar{G}_1, \bar{G}_2$  = Laplace transformations of  $G_1$  and  $G_2$   
 $h$  = height of cylindrical soil specimen  
 $I_0(\lambda), I_1(\lambda)$  = deformed Bessel functions of order zero and one  
 $J_1, J_2$  = creep functions in shear and in isotropic compression  
 $J_0(\nu), J_1(\nu)$  = Bessel functions of order zero and one  
 $k$  = coefficient of permeability  
 $n, \dot{n}, n_0$  = porosity, its time rate and initial porosity  
 $p, q$  = radial and axial compressive pressure  
 $r$  = nondimensional cylindrical coordinate  
 $\mathbf{r}$  = position vector  
 $s_{ij} = ij$  component of deviatoric stress tensor  
 $s_n$  = value of  $s$  corresponding to  $\nu_n$   
 $t$  = time  
 $T$  = time factor  $\left( \frac{k(2G_1 + G_2)}{3a^2\alpha} t \right)$   
 $T_{ij}'$  = Laplace transformation of  $\bar{\tau}_{ij}'$   
 $u$  = pore water pressure  
 $u_e, \bar{u}_e$  = excess pore water pressure and its nondimensional form  
 $u_i, \bar{u}_i$  =  $i$  component of displacement vector of soil skeleton and its nondimensional form  
 $u_{s,i}$  = settlement of upper side of soil specimen immediately after loading  
 $\mathbf{u}, \dot{\mathbf{u}}$  = displacement vector of soil skeleton and its time rate  
 $\mathbf{U}, \dot{\mathbf{U}}$  = displacement vector of pore water and its time rate  
 $V_i$  = Laplace transformation of  $\bar{u}_i$   
 $z$  = nondimensional cylindrical coordinate  
 $\alpha = (2 - n_0) \gamma_w$   
 $\gamma, \dot{\gamma}$  = weight of soil-water system per unit volume and its time rate  
 $\gamma_s, \gamma_w$  = weights of soil grains and of water per unit volume  
 $\varepsilon_{ij} = ij$  component of deviator strain tensor of soil skeleton  
 $\Theta_n = (2G_1 + G_2)^2 \nu_n^2 - 8G_1G_2$   
 $A = (2\bar{G}_1 + \bar{G}_2)\lambda I_0(\lambda) - 4\bar{G}_1 I_1(\lambda)$

$$\Pi = (2G_1 + G_2)\nu J_0(\nu) - 4G_1 J_1(\nu)$$

$\tau_{ij} = ij$  component of total stress tensor

$\tau_{ij}', \bar{\tau}_{ij}' = ij$  component of effective stress tensor ( $\tau_{ij}' = \bar{\tau}_{ij}'$ )

$\Phi$  = Laplace transformation of  $\bar{u}_e$

\* = notation representing Stieltjes convolution

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