ON SEQUENTIAL COMPACTNESS IN $(L_\varphi^*)_r$

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ABSTRACT

The precompact and complete space is sequentially compact in metric space. The converse of this assertion is also true. We define naturally r_precompact in the sense of ranked spaces. We prove that a subset of ranked space $(L_\varphi^*)\ r$ is sequentially compact provided that it is r_precompact.

§1. INTRODUCTION

The theory of ranked space, a revolutionary constructive method of the mathematical analysis, is introduced by Professor Kinjiro Kunugi [1]. The purpose of this concept is that it includes all of nonmetrizable function spaces and as far as possible properties of metric spaces. For instance, nuclear spaces and topological linear spaces are given in the sense of distributions. After, many results in the sense of ranked spaces are obtained in [5], [6] and [7].

A study of the relation between the compactness and completeness in ranked spaces is discussed in [4]. One of the results in [4] is that an r-compact ranked space satisfying the axiom (R_3) is compact, where the axiom (R_3) means the relation of $U(x) \supset cl(V(x))$ holds if $m<n$, $U(x) \in \mathcal{V}m(x)$, $V(x) \in \mathcal{V}n(x)$ and $U(x) \supset V(x)$.

In metric space, sequentially compact set is complete. But this assertion does not always hold in ranked spaces [4]. The Orlicz space L_φ^* is treated as a ranked space denoted by $(L_\varphi^*)_r$, and $(L_\varphi^*)_r$ is complete [2]. Precompact and complete space is sequentially compact in a metric space. In the same time the assertion of converse is also true.

In ranked spaces, certain properties of sequentially compact sets are obtained in [3]. One of results is that for any subset A of the ranked space X, the following three conditions are equivalent:

a) A is r-compact.

b) For any countable family $\{B_n\}$ of subsets of A,

with the finite intersection property, there is a $\tau \in T$ such that we have $A \cap (\bigcap_n cl_\tau(B_n)) \neq \phi$.

c) For any countable family $\{C_n\}$ of subsets of X and for every $\tau \in T$, $\{in_\tau(\check{C}_n)\}$ covers A, there is a finite subfamily of $\{C_n\}$ covering A.

In this paper, we define r_precompact in the sense of ranked space $(L_\varphi^*)_r$, from which we prove that a subset of $(L_\varphi^*)_r$ is sequentially compact in virtue of completeness of $(L_\varphi^*)_r$, $(L_\varphi^*)_r$ becomes a linear space [2]. Hence its indicator is ω_0.

§2. PRELIMINARIES

We consider a metric space (X,ρ), where ρ is said metric and hereafter is ommitted for brief expressions. A sequence $\{x_j\}$ in X is said to be Cauchy sequence if for arbitrary positive number ε there exists j_0 such that $\rho(x_j,x_k)<\varepsilon$ for all $j,k\geqq j_0$. When every Cauchy sequence $\{x_j\}$ in X converges to some x_0 in X with respect to ρ, X is said to be complete. X is said to be sequentially compact if any sequence $\{x_j\}$ in X contains subsequence $\{x_{j_k}\}$ which converges in X. The ε-neighborhood of x for positive number ε is denoted by $W(x\,;\,\varepsilon)$. A space X is said to be precompact if for arbitrary positive number ε, there exists correspondingly a set consisting of finite elements $x_1,x_2,\cdots,x_n$ in X such that:

$$W(x_1\,;\,\varepsilon)\cup W(x_2\,;\,\varepsilon)\cup\cdots\cup W(x_n\,;\,\varepsilon)=X,$$

namely, the finite union of $\{W(x_i\,;\,\varepsilon)\,;\,i=1,2,\cdots,n\}$ covers X. X is said to be compact if each open cover of X has a finite subcover.

Compactness is topological property. The concept of completeness and precompactness is uniformly topological properties but not topological one. X is precompact if, and only if, any sequence in $\{x_j\}$ in X always includes subsequence $\{x_{j_k}\}$ of $\{x_j\}$, where $\{x_{j_k}\}$ is Cauchy sequence. Furthermore, the following three conditions are equivalent in a metric space:

a) X is compact.
b) X is sequentially compact.
c) X is complete and precompact.

§3. CONCEPT OF RANKED SPACE.

N is a set of non negative integer, i.e. N ={0,1,2, ...}. Let X be non empty set and x be an element of X. If x belongs to subset V of X, V is said to be preneighborhood of x. This fact is denoted by $V\in\mathcal{V}(x)$ and we set $\mathcal{V}=\cup\{\mathcal{V}(x)\,:\,x\in X\}$. A space X is said to be ranked space, if it satisfies following two properties:

(a) $\forall V\in\mathcal{V}(x)\,:\,x\in V$,
(b) $\forall x\in X, \forall V\in\mathcal{V}(x), \forall n\in N, \exists k(\geqq n)\in N, \exists U\in\mathcal{V}(x)\cap\mathcal{V}_k\,:\,U\subset V$,
where $\mathcal{V}_k$ is associated with a subfamily of $\mathcal{V}$ for each $k\in N$, and is said to be rank k.

§4. DEFINITIONS AND LEMMAS.

In this section we refer to [2] for all definitions, terminologies and lemmas. Function φ is called an N-function if it admits of the representation

$$\varphi(u)=\int_0^{|u|}p(t)\,dt$$

where the function $p(t)$ is right-continuous for $t\geq 0$, positive for $t>0$, and non-decreasing which satisfies the conditions: $p(0)=0$, $p(\infty)=\lim_{t\to\infty}p(t)=\infty$. Let Φ be a family of N-functions. When f is a real valued function, we set

$\varphi(aL)=\{f : \int_{-\pi}^{\pi}\varphi(a|f(x)|)dx<\infty\}$ for $a>0$.

Definition 1. We define $L_\varphi^*=\bigcup_{a>0}\varphi(aL)$, where L_φ^* is called Orlicz space.

We set $X_\varphi^m=\varphi(2^{-m}L)$, for $m\in$ N and $D_\varphi^m(f)=\int_{-\pi}^{\pi}\varphi(2^{-m}|f(x)|)dx$ for $f\in X_\varphi^m$.◇

Definition 2. We set $V(f\ ;\ m,\varepsilon)=\{g\in X_\varphi^m : D_\varphi^m(f-g)<\varepsilon\}$ for $f\in X_\varphi^m$, $m\in N$, $\varepsilon>0$.◇

$V(f : m,2^{-n})$ is said to be preneighborhood of rank n of center f.

We set $\mathcal{V}^m(f)=\{V(f\ ;\ m\ ;\ \varepsilon) : \varepsilon>0\}$ for $f\in X_\varphi^m, m\in$ N and $\mathcal{V}_n^m=\{V(f\ ;\ m,2^{-n}) : f\in X_\varphi^m\}$ for n, $m\in$ N.

Definition 3. We define $\mathcal{V}(f)=\bigcup_{m\in M}\mathcal{V}^m(f)$ for $f\in L_\varphi^*$, $M=\{m\in N : f\in X_\varphi^m$, where $\mathcal{V}(f)$ is called a system of preneighborhoods of center f.◇

Definition 4. We define $\mathcal{V}_n=\bigcup_{m\in N}\mathcal{V}_n^m$.◇

Lemma 1. *Suppose* $f,g\in X_\varphi^m$, *then* $D_\varphi^{m-1}(f+g)\leqq D_\varphi^m(f)+D_\varphi^m(g)$.

Lemma 2. $L_\varphi^*=\bigcup_{m\in N}X_\varphi^m$ *is satisfied and it is a linear space.*

Lemma 3. L_φ^* *is a ranked space by a system of preneighborhoods introduced from definition* 2 *to definition* 4.

We denote by $(L_\varphi^*)_r$ as a ranked space to L_φ^*

Definition 5. A sequence of preneighborhoods $\{V(f\ ;\ m_i,2^{-n_i}) : i\in N\}$ in the ranked space $(L_\varphi^*)_r$ is called a fundamental sequence of center f, if the following two properties are satisfied:

(a) $V(f\ ;\ m_0,2^{-n_0})\supset V(f\ ;\ m_1,2^{-n_1})\supset\cdots$,

(b) $n_0\leqq n_1\leqq\cdots\ \lim_{i\to\infty}n_i=\infty$.◇

Definition 6. A sequence of functions $\{f_j\}$ in the ranked space $(L_\varphi^*)_r$ is said to be Cauchy sequence in $(L_\varphi^*)_r$ if there exists a fundamental sequence $\{V(0\ ;\ m_i,2^{-n_i})\ ;\ i\in N\}$ such that for each $i\geqq0$, non negative integer j_0 can be found such that $f_k-f_l\in V(0\ ;\ m_i,2^{-n_i})$ for all $k,l\geqq j_0$.◇

Lemma 4. A sequence of functions $\{f_j\}$ *in the ranked space* $(L_\varphi^*)_r$ *is Cauchy sequence in* $(L_\varphi^*)_r$ *if, and only if, there exists an* $m\in$ N *such that* $\lim_{k,l\to\infty}\int_{-\pi}^{\pi}\varphi(2^{-m}|f_k(x)-f_l(x)|)dx=0$.

Lemma 5. $(L_\varphi^*)_r$ *is complete.*

Proofs of above lemmas are in [2].

§5. SEQUENTIALLY COMPACT.

Now we define the concept of sequentially compact and r_precompact in the sense of ranked space $(L_\varphi^*)_r$

Definition 7. We define that a subset A of $(L_\varphi^*)_r$ *is* sequentially compact i.e. any sequence $\{f_j\}$ in A contains subsequence $\{f_{j_k}\}$ which converges in A.◇

Definition 8. Let A be a subset of ranked space $(L_\varphi^*)_r$. A is said to be r_precompact if for any $\varepsilon>0$, there exists an $m \in \mathrm{N}$ and a finite set of $\{g_1,\cdots,g_n\}$ in A such that:

$$V(g_1, m_1, \varepsilon) \cup V(g_2, m_2, \varepsilon) \cup \cdots \cup V(g_n, m_n, \varepsilon) \supset A. \diamond$$

Theorem. Let a subset A of $(L_\varphi^*)_r$ r_precompact. *Then A is sequentially compact.*

Proof. Let $\{f_j\}$ be an arbitrary sequence of functions in A. Since A is r_precompact, the following is satisfied:

$$V(g_1^{(1)}, m_1^{(1)}, 2^{-1}) \cup \cdots \cup V(g_{n1}^{(1)}, m_{n1}^{(1)}, 2^{-1}) \supset A.$$

At least one of these V's contains infinite $\{f_{jk}\}$ subsequence of $\{f_j\}$, to which we denote by V_1 and $f_{1,1}, f_{1,2}, \cdots$ arranged in increasing order of $\{f_{j_k}\}$. Put $m^{(1)} = max\{m_1^{(1)}, m_2^{(1)}, \cdots, m_{n_1}^{(1)}\}$, then $V_1 \subset V(g^{(1)}, m^{(1)}, 2^{-1})$. By repeating the same way, we can find V_k, which contains $\{f_{k,1}, f_{k,2}, \cdots\}$ which is subsequence of $\{f_{k-1,1}, f_{k-1,2}, \cdots\}$ and $V_k \subset V(g^{(k)}, m^{(k)}, 2^{-k}\}$, where $m^{(k)} = max\{m_1^{(k)}, m_2^{(k)}, \cdots, m_{n_k}^{(k)}\}$. For each $\varepsilon > 0$, there exists $j_0 \in \mathrm{N}$ such that $2^{-j_0} < \varepsilon/2$. For all $k, l > j_0$, $V_{j_0} = V(g\ ;\ m, 2^{-j_0})$ contains, $f_{k,k}, f_{l,l}$, i.e. $D_\varphi^m(g - f_{k,k}) < 2^{-j_0}, D_\varphi^m(g - f_{l,l}) < 2^{-j_0}$, where $m = max\{m_1^{(j_0)}, m_2^{(j_0)}, \cdots, m_{n_{j_0}}^{(j_0)}\}$. By lemma 1, therefore we obtain the following:

$$\begin{aligned}
&\int_{-\pi}^{\pi} \varphi(2^{-m-1}|f_{k,k}(x) - f_{l,l}(x)|)dx \\
&\leqq \int_{-\pi}^{\pi} \varphi(2^{-m-1}[|f_{k,k}(x) - g(x)| + |g(x) - f_{l,l}(x)|])dx \\
&= D_\varphi^{m+1}[(f_{k,k} - g) + (g - f_{l,l})] \\
&\leqq D_\varphi^m(f_{k,k} - g) + D_\varphi^m(g - f_{l,l}) \\
&< 2 \cdot 2^{-j_0} < \varepsilon.
\end{aligned}$$

Put $f_{j_k} = f_{k,k}$ Then $\{f_{j_k}\}$ is subsequence of $\{f_j\}$ and also Cauchy sequence in $(L_\varphi^*)_r$. In virtue of lemma 5, A is sequentially compact.

Q.E.D

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